G.P.Ismatullaev, M.S.Kosbergenova

HISOBLASH USULLARI

Toshkent – 2014

O'ZBEKISTON RESPUBLIKASI
OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI

G.P.Ismatullayev, M.S. Kosbergenova

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O'zbekiston Respublikasi Oliy va o'rta maxsus ta'lim vazirligi tomonidan oliy o'quv yurtlarining 5130100 – «Matematika», 5130200 – «Amaliy matematika va informatika», 5140300 – «Mexanika», 5330100 – «Axborot tizimlarining matematik va dasturiy ta'minoti», 5330200 – «Informatika va axborot texnologiyalari» hamda 5330300 – «Axborot xavfsizligi» ta'lim yo'nalishlari bo'yicha tahsil olayotgan talabalar uchun o'quv qo'llanma sifatida tavsiya etilgan

«TAFAKKUR-BO'STONI»
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Ushbu o'quv qo'llanmada hisoblash usullarining xatoliklar nazariyasi, funksiyalarni yaqinlashtirish, aniq integrallarni taqribiy hisoblash, algebraik va transsendent tenglamalarni, chiziqli algebraik tenglamalar sistemasini sonli yechish usullari, matritsaning xos son va vektorlarini taqribiy topish usullari, Koshi va chegaraviy masalalarni taqribiy yechish, matematik fizika tenglamalariga qo'yilgan chegaraviy masalalarni to'r usuli bilan yechish kabi usullari batafsil yoritilgan. Bayon qilingan har bir usulga oid misollar yechib ko'rsatilgan. Shuningdek, hisoblash usullari amaliyoti uchun har bobga tegishli misollar to'plami ham berilgan.

O'quv qo'llanma oliy o'quv yurtlarining matematika, amaliy matematika va informatika, mexanika, axborot tizimlarining matematik va dasturiy ta'minoti, informatika va axborot texnologiyalari hamda axborot xavfsizligi yo'nalishlari bo'yicha tahsil olayotgan talabalari uchun mo'ljallangan.

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I BOB. XATOLIK NAZARIYASI

1.1-§. Xatoliklar manbayi va klassifikatsiyasi

Ixtiyoriy matematik masalani sonli yechishda biz aniq yechimga ega bo‘lmasdan, balki yechimni u yoki bu darajadagi aniqlikda topamiz. Demak, natijadagi xatolik qanday hosil bo‘lganligini aniqlash lozimdir.

Har qanday matematik masalaning qo‘yilishida turli miqdorlar (parametrlar) qatnashadi. Bizni qiziqtirgan yechimni topishimiz uchun masalada qatnashuvchi parametrlarning qiymati berilgan bo‘lishi kerak.

Misol uchun,

$$y' = f(x, y)$$

differensial tenglamaning konkret xususiy yechimini topish uchun $y(x_0) = y_0$ – boshlang‘ich shart berilgan bo‘lishi kerak. Bundan tashqari, differensial tenglamaning o‘ng tomoni $f(x, y)$ ba’zi parametrlarga bog‘liq bo‘lsa, ularning qiymatlari ham berilgan bo‘lishi shart. Berilgan masalaning bizni qiziqtirgan yechimini topish uchun, masalada qatnashuvchi barcha parametrlarni *dastlabki ma’lumotlar* deb ataymiz. Tabiiyki, topilishi kerak bo‘lgan (bizni qiziqtirgan) yechim dastlabki ma’lumotlarning funksiyasi bo‘ladi. Ko‘pincha dastlabki ma’lumotlar yoki tajribadan, yoki biron-bir boshqa masalani yechishdan hosil bo‘ladi. Har ikki holda ham dastlabki ma’lumotlarning aniq qiymatiga emas, balki uning taqribiy qiymatiga ega bo‘lamiz. Shuning uchun dastlabki ma’lumotlarning berilgan har bir qiymati uchun masalani aniq yechganimizda ham, baribir, taqribiy natijaga ega bo‘lamiz va natijaning aniqligi dastlabki ma’lumotlarning aniqligiga bog‘liq bo‘ladi.

Aniq yechim bilan taqribiy yechim orasidagi farq *xato* deyiladi. Dastlabki ma’lumotlarning noaniqligi natijasida hosil bo‘lgan xato *yo‘qotilmas xato* deyiladi. Tabiiyki, bu xatolik berilgan masalani yechuvchiga bog‘liq bo‘lmay, to‘la-to‘kis berilgan ma’lumotlarning aniqligiga bog‘liqdir. Agar boshlang‘ich ma’lumotlarning aniqligi ma’lum bo‘lsa, matematik masala yechimining xatoligini baholay bilish kerak.

Ma’lumki, ba’zi matematik ifodalar tabiat hodisalarining ozmi-ko‘pmi ideallashtirilgan modelini tasvirlaydi. Shuning uchun tabiat hodisalarining aniq matematik ifodasini (tenglamalarini, formulasini) berib bo‘lmaslik bois, natijada xato kelib chiqadi. Bundan tashqari, biron masala aniq matematik formada yozilgan bo‘lsa va uni bu ko‘rinishda yechish mumkin bo‘lmasa, u holda bu masala unga yaqinroq va yechish mumkin bo‘lgan masalaga almashtiriladi. Buning natijasida hosil bo‘ladigan xato *metod xatoligi* deyiladi.

Boshlang‘ich berilgan masala sonli yechilishi mumkin bo‘lgan masalaga almashtirilgan bo‘lsa va, hatto, boshlang‘ich qiymatlar aniq bo‘lsa ham aniq yechimga ega bo‘la olmaymiz. Bu holat quyidagicha izohlanadi: birinchidan, masalada turli irratsional sonlar qatnashishi mumkin, tabiiyki, ularni taqribiy qiymatlariga almashtiramiz; ikkinchidan, hisoblash jarayonida oraliq natijalar yaxlitlanadi. Hisoblashlar jarayonida hosil bo‘ladigan xatolik *hisoblash xatoligi* deyiladi.

Shunday qilib, yechimning *to‘liq xatoligi*, ya’ni berilgan masalaning aniq yechimi bilan amalda topilgan taqribiy yechim orasidagi farq yo‘qotilmas xato, metod xatoligi va hisoblash xatoligidan iborat bo‘lar ekan.

1.2-§. Absolut va nisbiy xatolar

Agar A – biror miqdorning aniq qiymati bo'lib, a uning ma'lum taqribiy qiymati bo'lsa, u vaqtda a sonning *absolut xatoligi* deb $\Delta a = |A - a|$ ga aytiladi. Absolut xatolik faqat nazariy ahamiyatga egadir, chunki ko'pincha biz A ning qiymatini bilmaymiz, shuning uchun Δa ni ham bilmaymiz. Lekin Δa ning o'zgarish chegaralarini ko'rsatishimiz mumkin. Bu chegaralar taqribiy a sonni topish usuli bilan aniqlanadi. Masalan, biz o'lchashni oddiy chizg'ich bilan bajarsak, absolut xatolik, odatda, $0{,}5$ mm dan oshmaydi, agarda shu ishni shtangensirkulda bajarsak, absolut xatolik $0{,}1$ mm dan oshmaydi.

Absolut xatodan kichik bo'lmagan har qanday songa taqribiy a sonning *absolut limit xatosi* $\Delta(a)$ deb aytiladi. Bu ta'rifdan $|A - a| \leq \Delta(a)$, bundan esa $a - \Delta(a) \leq A \leq a + \Delta(a)$ kelib chiqadi.

Absolut xato va limit absolut xato hisoblash xatoligini baholash uchun yetarli emas. Misol uchun, ikkita og'irlik o'lchanganda $m_1 = 100{,}2\ \text{g} \pm 0{,}2$ g va $m_2 = 12{,}6 \pm 0{,}2$ g natijalar hosil bo'lsin, bu yerda har ikkalasida limit absolut xatolik bir xil bo'lishidan qat'i nazar birinchi o'lchash ikkinchi o'lchashdan ancha aniqdir. Aniqlikni yaxshiroq baholaydigan tushuncha kiritamiz.

Absolut xatoning taqribiy sonning absolut qiymatiga nisbati taqribiy sonning *nisbiy xatosi* δa deb aytiladi:

$$\delta a = \frac{\Delta a}{|a|}.$$

Xuddi yuqoridagidek *limit nisbiy xato* $\delta(a)$ tushunchasi kiritiladi:

$$\delta(a) = \frac{\Delta(a)}{|a|}.$$

Limit nisbiy xatolik yordamida A son quyidagicha yoziladi:

$$A=a(1\pm\delta(a)).$$

Bundan keyin biz limit absolut xato va limit nisbiy xatoni qisqacha absolut va nisbiy xato deymiz. Absolut xato ismli, nisbiy xato ismsiz miqdordir. Odatda, nisbiy xato protsentlarda yoziladi.

Sonning ifodasidagi (yozilishidagi) chap tomondan birinchi noldan farqli raqamidan boshlab barcha raqamlar va saqlanilgan razryadlarni bildiruvchi oxirgi nollar taqribiy sonning *ma'noli raqamlari* deyiladi.

Agar $\Delta a\le\frac{1}{2}10^{m-n+1}$ tengsizlik bajarilsa, u holda taqribiy

$$a=\alpha_m\cdot 10^m+\alpha_{m+1}\cdot 10^{m-1}+\cdots+\alpha_{m+n-1}\cdot 10^{m-n+1}+\cdots,$$

sonning birinchi n ta ma'noli raqami *ishonchli raqamlar* deyiladi.

Taqribiy son a ning ishonchli raqamlari soni bilan uning nisbiy xatoligi orasida quyidagi

$$\delta a\le\frac{1}{\alpha_m}\left(\frac{1}{10}\right)^{n-1}$$

bog'lanish mavjud.

Isboti. Taqribiy a son n ta ishonchli raqamga ega bo'lganligi uchun uning ko'rinishi quyidagicha bo'ladi:

$$a=\alpha_m\cdot 10^m+\alpha_{m+1}\cdot 10^{m-1}+\cdots+\alpha_{m+n-1}\cdot 10^{m-n+1}+\cdots,$$

bu yerda $\alpha_m\ge 1$.

$$\Delta a=\frac{1}{2}10^{m-n+1}$$

bo'lib, bundan

$$A\ge a-\frac{1}{2}10^{m-n+1}$$

bo'ladi.

a ni undan katta bo'lmagan $\alpha_m\cdot 10^m$ songa almashtirsak, bu tengsizlik yanada kuchayadi:

$$A \geq \alpha_m \cdot 10^m - \frac{1}{2} 10^{m-n+1} = \frac{1}{2} 10^m \left(2\alpha_m - \frac{1}{10^{n-1}} \right) \qquad (1)$$

Bu tengsizlikning o'ng tomoni $n = 1$ da eng kichik qiymatga ega bo'ladi, shuning uchun

$$A \geq \frac{1}{2} 10^m (2\alpha_m - 1), \qquad (2)$$

bo'ladi. $2\alpha_m - 1 = \alpha_m + (\alpha_m - 1) \geq \alpha_m$ ligidan

$$A \geq \frac{1}{2} \alpha_m 10^m$$

bo'lib

$$\delta a = \frac{\Delta a}{|a|} \leq \frac{\frac{1}{2} 10^{m-n+1}}{\frac{1}{2} \alpha_m \cdot 10^m} = \frac{1}{\alpha_m} \left(\frac{1}{10} \right)^{n-1}$$

ta'kid o'rinliligi kelib chiqadi.

Natija 1. Taqribiy a sonining limit nisbiy xatoligi deb

$$\delta(a) = \frac{1}{\alpha_m} \left(\frac{1}{10} \right)^{n-1} \qquad (3)$$

ni olish mumkin, bu yerda $\alpha_m \neq 0$ bo'lib, a taqribiy sonning birinchi ma'noli raqami.

Natija 2. Agar taqribiy a sonning ishonchli raqamlar soni ikkidan katta bo'lsa, amaliyotda

$$\delta(a) = \frac{1}{2\alpha_m} \left(\frac{1}{10} \right)^{n-1} \qquad (4)$$

deb olish o'rinli.

Haqiqatan, (1) da ishtirok etuvchi $\frac{1}{10^{n-1}}$ sonni e'tiborga olmasak ham bo'ladi. U holda

$$A \geq \frac{1}{2} \cdot 10^m \cdot 2 \cdot \alpha_m = \alpha_m \cdot 10^m$$

bo'lib, bundan

$$\delta(a) \leq \frac{\frac{1}{2} 10^{m-n+1}}{\alpha_m \cdot 10^m} = \frac{1}{2\alpha_m}\left(\frac{1}{10}\right)^{n-1}$$

bo'ladi.

1.3-§. Funksiya xatoligi

Faraz qilaylik, uzluksiz differensiallanuvchi

$$u = f(x_1, x_2, ..., x_n)$$

funksiya argumentlarining taqribiy qiymatlari $\bar{x}_i$ va ularning absolut xatoliklari $\Delta \bar{x}_i$, $i = 1,2,...,n$ ma'lum bo'lsin. Berilgan funksiyaning limit absolut $\Delta \bar{u}$ va limit nisbiy $\delta \bar{u}$ xatoligini topishni ko'raylik.

Bu masalani hal etish uchun quyidagi shartlar o'rinli bo'lishligini talab etamiz:

1. Qaralayotgan sohada f uzluksiz differensiallanuvchi bo'lib, xususiy hosilalari sekin o'zgaruvchi bo'lsin.

2. Argumentlarning absolut xatoliklari aytarli katta emas, nisbiy xatoliklari yetarlicha kichik bo'lsin.

U holda Lagranj formulasiga ko'ra

$$u - \bar{u} = f(x_1, x_2, ..., x_n) - f(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) = \sum_{i=1}^{n} \frac{\partial f(\xi)}{\partial x_i} \cdot (x_i - \bar{x}_i),$$

bu yerda $\xi = (\xi_1, \xi_2, ..., \xi_n)$ esa $(x_1, x_2, ..., x_n)$ va $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ nuqtalarni birlashtiruvchi kesmaga tegishli qandaydir nuqta.

Funksiyaga qo'yilgan birinchi shartlarga asosan, $\frac{\partial f(\xi)}{\partial x_i}$ ni $\frac{\partial f(\bar{x})}{\partial x_i}$ ga almashtirish mumkin:

$$u - \bar{u} = \sum_{i=1}^{n} \frac{\partial f(\bar{x})}{\partial x_i} \cdot (x_i - \bar{x}_i).$$

Bundan

$$\Delta(\bar{u})=\sum_{i=1}^{n}\left|\frac{\partial f(\bar{x})}{\partial x_i}\right|\cdot\Delta(\bar{x}_i).$$

Endi nisbiy xatoligini chiqaramiz:

$$\delta(\bar{u})=\frac{\Delta(\bar{u})}{|f(\bar{x})|}=\sum_{i=1}^{n}\left|\frac{\frac{\partial f(\bar{x})}{\partial x_i}}{f(\bar{x})}\right|\cdot\Delta(\bar{x}_i).$$

Bu formulaning quyidagicha ko'rinishi ham ishlatishadi:

$$\delta(\bar{u})=\sum_{i=1}^{n}\left|\frac{\frac{\partial f(\bar{x})}{\partial x_i}}{f(\bar{x})}\right|\cdot\Delta(\bar{x}_i)=\sum_{i=1}^{n}|x_i|\cdot\left|\frac{\frac{\partial f(\bar{x})}{\partial x_i}}{f(\bar{x})}\right|\cdot\delta(\bar{x}_i).$$

Bobga tegishli tayanch iboralar: yo'qotilmas xato, metod xatoligi, hisoblash xatoligi, to'liq xatolik. Taqribiy son, absolut xatolik, nisbiy xatolik, limit absolut xatolik, limit nisbiy xatolik, ma'noli raqamlar, ishonchli raqamlar, funksiya xatoligi.

Savollar

1. Yo'qotilmas xato, metod xatoligi va hisoblash xatoligi tushunchalarini izohlang.
2. Hisoblash jarayonida hosil bo'ladigan to'liq xatolik nimalardan iborat bo'ladi?
3. Taqribiy sonning absolut, nisbiy xatoliklarini ta'rifini ayting.
4. Limit absolut xatolik va limit nisbiy xatolik ta'rifini ayting.
5. Taqribiy sonning ma'noli va ishonchli raqamlari ta'rifini ayting.
6. Taqribiy sonning ishonchli raqamlari soni bilan uning nisbiy xatoligi orasidagi bog'lanishni isbotlang.
7. Funksiyaning limit absolut va limit nisbiy xatoliklari formulasini chiqaring.

Misol 1. $y=\sin x$ funksiyaning x ning aniq qiymatidagi limit absolut xatoligini aniqlang.

Yechish. Ma'lumki, berilgan funksiyani x ning darajalari bo'yicha Teylor qatoriga yoyilmasi quyidagicha:

$$y=\sin x=\sum_{k=o}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}+...+(-1)^k\frac{x^{2k+1}}{(2k+1)!}+...$$

$y=\sin x$ ning taqribiy qiymatini hisoblash uchun bu qatorda chekli miqdorda hadlarini olish kifoyadir, ya'ni

$$y^*=(\sin x)^*=\sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}.$$

Bu yaqinlashuvchi qatorning absolut xatoligi

$$\left|y-y^*\right|\leq\frac{|x|^{2n+3}}{(2n+3)!}$$

ekanligi bizga ma'lumdir. Demak, limit nisbiy xatolik

$$\Delta(y)=\frac{|x|^{2n+3}}{(2n+3)!}$$

ga teng bo'ladi.

Misol 2. Quyida berilgan

$$\begin{cases}2x+y=0,\\10^{-5}x+y=1\end{cases}$$

tenglamalar sistemasini o'nlik sanoq sistemasida to'rtta raqam aniqligida amallarni bajaruvchi kompyuterda yeching.

Yechish. Qo'l ostimizdagi kompyuterda sonlar $0,x_1x_2x_3x_4\cdot10^p$ ko'rinishda ifodalanadi, bu yerda: $0\leq x_i\leq9$, $i=1,2,3,4$. Berilgan sistemaning aniq yechimi:

$$x=-\frac{100000}{200001}\approx-0,4999975,\qquad y=\frac{200000}{200001}\approx0,999995.$$

Ikkinchi tenglamadan

$$x = 10^5 y - 10^5$$

ga ega bo'lamiz. Birinchi tenglamadan $(2\cdot 10^5 + 1)y = 2\cdot 10^5$ ekanligi kelib chiqadi yoki $0,200001\cdot 10^6 y = 0.2\cdot 10^6$ bo'ladi. Noma'lum y oldidagi koeffitsiyent bizning kompyuterda $0,2000\cdot 10^6$ ko'rinishda ifodalanadi. Hisoblashlardagi yaxlitlash evaziga shu ko'rinishga ega bo'lamiz, natijada $y = 0,1\cdot 10^1 = 1$ bo'lib, $x = 0$ ga ega bo'lamiz.

Ishlatilgan algoritm aqlbovar qilmaydigan xatoliklarga olib kelsa, bunday algoritmlar *sonli noturg'un* deb ataladi.

Misol 3. Agar π soni o'rniga 3,14 desak, limit nisbiy xatolik qanday bo'ladi?

Yechish. $\alpha_m = 3$ va $n = 3$ bo'lganligi uchun (4) formulaga ko'ra,

$$\delta(\pi) = \frac{1}{2\cdot 3}\left(\frac{1}{10}\right)^{3-1} = \frac{1}{600}$$

bo'ladi.

Misol 4. $a = \sqrt{22}$ da nechta raqam olsak, nisbiy xatolik $\delta a = 0,001$ bo'ladi?

Yechish. $\delta a \le \frac{1}{\alpha_m}\left(\frac{1}{10}\right)^{n-1}$ formuladan foydalanamiz. $a = \sqrt{22}$ ning birinchi raqami $\alpha_m = 4$ va $\delta a = 0,001$ bo'lganligi uchun

$$\frac{1}{4\cdot 10^{n-1}} \le 0,001$$

bo'lib, bundan $10^{n-1} \ge 250$, $n \ge 4$ bo'ladi.

Agar $\sqrt{22}$ da 4 ta raqam olsak, uning nisbiy xatoligi 0,001 bo'ladi.

Misol 5. $a = 24253$ taqribiy sonning nisbiy xatoligi 0,1% bo'lsa, uning nechta raqami ishonchli bo'ladi?

Yechish. $\delta a = \frac{0{,}1\%}{100\%} = 0{,}001$, $\Delta a = a \cdot \delta a = 24253 \cdot 0{,}001 \cong 24{,}3 = 2{,}43 \cdot 10$,

$\Delta a \le \frac{1}{2} 10^{m-n+1}$ bo'lsa, a sonining birinchi n ta ma'noli raqami ishonchli bo'lar edi. Bu yerda $m=4$ bo'lganligi uchun $2{,}43 \cdot 10 \le \frac{1}{2} 10^{4-n+1}$ dan $n=3$ desak bo'ladi. Berilgan sonni $a = 243 \cdot 10^2$ deb yozish maqsadga muvofiqdir.

Misol 6. Agar $x_1 = 12{,}2$ va $x_2 = 73{,}56$ bo'lib, ulardagi barcha raqamlar ishonchli bo'lsa, $U = x_1 \cdot x_2$ ko'paytmada nechta ishonchli raqam bo'ladi?

Yechish.

$$\Delta(x_1) = \frac{1}{2} 10^{1-3+1} = 0{,}05,\ \Delta(x_2) = \frac{1}{2} 10^{1-4+1} = 0{,}005.$$

Bundan, $\delta(u) = \frac{\Delta(x_1)}{x_1} + \frac{\Delta(x_2)}{x_2} = \frac{0{,}05}{12{,}2} + \frac{0{,}005}{73{,}56} = 0{,}0042$, $u = 897{,}432$ bo'lib, $\Delta(u) = u \cdot \delta(u) = 897{,}432 \cdot 0{,}0042 = 3{,}6$ bo'ladi. $\Delta u \le \frac{1}{2} 10^{m-n+1}$ tengsizlikda $m = 2$ bo'lib, $n = 2$ ekanligi kelib chiqadi. Demak u kamida 2 ta ishonchli raqamga ega va uni $u = 874 \pm 4$ deb yozish maqsadga muvofiqdir.

Misol 7. Agar shar diametri $d = 3{,}7\,\text{sm} \pm 0{,}5\,\text{sm}$ bo'lsa, shar hajmi $V = \frac{1}{6}\pi d^3$ ning limit absolut va limit nisbiy xatoligi qanday bo'ladi?

Yechish. $\frac{\partial V}{\partial \pi} = \frac{1}{6} d^3 = 8{,}44$, $\frac{\partial V}{\partial d} = \frac{1}{2}\pi d^2 = 21{,}5$.

$$\Delta(V)=\left|\frac{\partial V}{\partial \pi}\right|\cdot|\Delta\pi|+\left|\frac{\partial V}{\partial d}\right|\cdot|\Delta d|=8,44\cdot 0,0016+21,5\cdot 0,05=$$

$$=0,013+1,075=1,088,\ \ \Delta(V)\cong 1,1\ \ \text{sm}^3.$$

Demak, $V=\frac{1}{6}\pi d^3=27,4\,\text{sm}^3\pm 1,1\,\text{sm}^3$,

$$\delta(V)=\frac{1,088}{27,4}=0,0397\cong 0,04,\ \ \delta(V)=4\%.$$

Javob: $\Delta(V)=1,1\,\text{sm}^3$, $\delta(V)=4\%$.

Misol 8. To'g'ri to'rtburchakning tomonlari $a\cong 5\,\text{m}$, $b\approx 200\,\text{m}$ bo'lsin. $\Delta(S)=1\,\text{m}^2$ dan oshmasligi uchun $\Delta(a)=\Delta(b)$ qanday bo'lishi kerak?

Yechish.

$$S=a\cdot b,\ \Delta(S)=b\cdot\Delta(a)+a\cdot\Delta(b)=\Delta(a)(a+b).$$

$$\Delta(a)=\frac{\Delta(S)}{a+b}=\frac{1}{205}<0,005,\ \Delta(a)=5\,\text{mm}.$$

Javob: $\Delta(a)=\Delta(b)=5$ mm.

Misollar.

$U=f(x_1,x_2,x_3)$ funksiyada $x_1=5,48$; $x_2=2,45$; $x_3=0,863$ bo'lib, $\Delta x_1=0,02$; $\Delta x_2=0,01$; $\Delta x_3=0,004$ bo'lsa, U funksiyaning absolut, nisbiy xatoliklarini va uning limit absolut, limit nisbiy xatoliklarini hamda ishonchli raqamlarini toping.

1. $U=\frac{x_1\cdot x_2^3}{48x_3}$.
2. $U=\sqrt{\frac{x_1\cdot x_2}{x_3}}$.
3. $U=\frac{x_1^2\cdot x_2}{48x_3^3}$.
4. $U=\frac{x_1^2\cdot x_2\cdot x_3^2}{4}$.
5. $U=\frac{x_1\cdot x_2}{x_3^2}$.
6. $U=\frac{x_1^2\cdot x_2}{x_3}$.

7. $U = \frac{x_1 \cdot x_2}{\sqrt[3]{x_3}}$.

8. $U = \frac{\sqrt{x_1} \cdot x_2}{x_3}$.

9. $U = \frac{\sqrt{x_1 \cdot x_2}}{x_3}$.

10. $U = \sqrt{\frac{x_1 \cdot x_2}{x_3}}$.

11. $U = \frac{(x_1 + x_2)x_3}{x_2 - x_3}$.

12. $U = \frac{x_1^3(x_2 + x_3)}{x_1 - x_2}$.

13. $U = \frac{\sqrt{x_1^2 - x_2^2}}{x_3}$.

14. $U = \frac{x_1 + x_2^2}{(x_2 - x_3)x_1}$.

15. $U = \frac{\sqrt{x_1 + x_2}}{\sqrt{x_3}}$.

16. $U = \frac{x_1(x_2 - x_3)}{\sqrt{x_3}}$.

17. $U = \frac{x_1 - x_2}{\sqrt{x_2 + x_3}}$.

18. $U = \frac{x_3\sqrt{x_1 - x_2}}{x_2 + x_3}$.

19. $U = \frac{(x_1 + x_2)^2}{2x_3}$.

20. $U = \frac{x_1 + x_2^2}{4x_3}$.

II BOB. FUNKSIYALARNI YAQINLASHTIRISH

Funksiyalarni yaqinlashtirish masalasining qo'yilishi

Funksiyalarni yaqinlashtirish masalasi qo'yilgan talabga (shartga) qarab turlicha bo'ladi. Hisoblash matematikasida keng qo'llaniladigan usullardan ba'zilarini eslatib o'tamiz. Xususan, interpolyatsiyalash, o'rtacha kvadratik ma'noda yaqinlashtirish, tekis yaqinlashish va splayn yaqinlashish. Interpolyatsiyalash funksiyalarni yaqinlashtirish nazariyasida olingan natijalarni funksiya jadvalini zichlashtirish, sonli differensiallash va integrallash, matematik fizika masalalarining to'rdagi analogini qurishda keng qo'llaniladi.

2.1-§. Algebraik interpolyatsiyalash masalasining qo'yilishi

$[a,b]$ oraliqda turli $n+1$ ta x_k, $k=0,1,...,n$ nuqtalarda $f(x)$ funksiyaning qiymatlari $f(x_k)$, $k=0,1,...,n$ berilgan bo'lsin. Darajasi n ga teng shunday

$$L_n(x)=a_0+a_1x+\cdots+a_nx^n \qquad (1)$$

algebraik ko'phad qurilsinki, u

$$a_0+a_1x_k+\cdots+a_nx_k^n=f(x_k),\ k=0,1,...,n \qquad (2)$$

shartlarni qanoatlantirsin.

(1) chiziqli algebraik tenglamalar sistemasining determinanti Vandermond determinantidir, u noldan farqli, chunki x_k, $k=0,1,...,n$ lar turli. Demak, (1) ko'phadning koeffitsiyentlari (2) dan bir qiymatli ko'rinishda topiladi.

$$L_n(x_i) = f(x_i), \quad i = 0,1,...,n \tag{3}$$

shartni qanoatlantiradigan $L_n(x)$ ko'phad $f(x)$ funksiyaning $\{x_i\}_{i=0}^{n}$ tugun nuqtalar yordamida qurilgan *interpolyatsion ko'phad* deyiladi.

Bu interpolyatsion ko'phadning ko'proq ishlatiladigan ko'rinishi – Lagranj formulasini keltiramiz. Uni quyidagi ko'rinishda yozamiz:

$$L_n(x) = \sum_{k=0}^{n} b_k(x) f(x_k) \tag{4}$$

(3) ga asosan

$$\sum_{k=0}^{n} b_k(x_i) f(x_k) = f(x_i), \quad i = 0,1,...,n$$

bo'ladi.

Bu munosabatlar o'rinli bo'lishi uchun $b_k(x)$ funksiyalar quyidagi shartlarni bajarishi kerak:

$$b_k(x_i) = \begin{cases} 0, & i \neq k \\ 1, & i = k, \; i = 0,1,...,n. \end{cases}$$

Bulardan ko'rinib turibdiki, har bir $b_k(x)$ $[a,b]$ da n tadan kam emas nolga ega bo'ladi. Biroq $L_n(x)$ darajasi n ga teng bo'lgan algebraik ko'phad bo'lganligi uchun $b_k(x)$ ning darajasini n ga teng ko'phad ko'rinishida izlash maqsadga muvofiqdir. Uni quyidagicha yozamiz:

$$b_k(x) = A_k(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n).$$

$b_k(x_k) = 1$ shartdan

$$A_k^{-1} = (x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)$$

kelib chiqadi. Agar $\omega_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$ deb olsak, $\omega_{n+1}'(x_k) = A_k^{-1}$ bo'ladi va (4) ko'phad

$$L_n(x) = \sum_{k=0}^{n} \frac{\omega_{n+1}(x)}{(x - x_k) \cdot \omega'_{n+1}(x_k)} f(x_k) \qquad (5)$$

ko'rinishga ega bo'ladi va uni *Lagranj interpolyatsion ko'phadi* deyiladi. (5) dagi $\frac{\omega_{n+1}(x)}{(x-x_k)\cdot\omega'_{n+1}(x_k)}$ ko'phad interpolyatsiyalashning *fundamental ko'phadlari* deyiladi, ba'zan uni x_k nuqtaning ta'sir etuvchi ko'phadi deb ham aytiladi.

2.2-§. Interpolyatsiyalash xatoligi

Biz $f(x)$ funksiyani interpolyatsion $L_n(x)$ ko'phadga almashtirganimizda

$$r_n(x) = f(x) - L_n(x),$$

xatolikka yo'l qo'yamiz. Bu *interpolyatsiyalash xatoligi* deyiladi. Tugun nuqtalarda xatolik nolga teng. $[a,b]$ ga tegishli ixtiyoriy x nuqtadagi ifodasini topamiz va baholaymiz. Buning uchun quyidagi funksiyani qaraymiz:

$$\varphi(z) = f(z) - L_n(z) - K\omega_{n+1}(z) \qquad (1)$$

bu yerda $z \in [a,b]$, K – o'zgarmas va

$$\omega_{n+1}(z) = (z - x_0)(z - x_1)\cdots(z - x_n). \qquad (2)$$

(1)dagi o'zgarmas K ni $\varphi(x) = 0$ shartdan topamiz:

$$K = \frac{f(x) - L_n(x)}{\omega_{n+1}(x)}. \qquad (3)$$

$f(z)$ funksiya $[a,b]$ da $n+1$ marta uzluksiz differensiallanuvchi bo'lsin deymiz. $\varphi(z)$ funksiya $[a,b]$ da $n+2$ ta nuqtada nolga teng, ular $x, x_0, x_1, \ldots, x_n$. Roll teoremasiga asosan, $\varphi'(z)$ $[a,b]$ ga tegishli $n+1$ ta, $\varphi''(z)$ n ta nolga ega bo'ladi va hokazo. $\varphi^{(n+1)}(z)$ $[a,b]$ da kamida bitta nolga ega bo'ladi, ya'ni $\varphi^{(n+1)}(\xi)=0, \xi\in[a,b]$. (1) dan $n+1$ marta hosila olib, $z=\xi$ desak, quyidagiga ega bo'lamiz:

$$f^{(n+1)}(\xi)=K\cdot(n+1)! \qquad (4)$$

(3) va (4) dan

$$f(x)-L_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot\omega_{n+1}(x) \qquad (5)$$

kelib chiqadi. Bundan

$$|f(x)-L_n(x)|\le\frac{M_{n+1}}{(n+1)!}|\omega_{n+1}(x)| \qquad (6)$$

bahoga ega bo'lamiz, bu yerda $M_{n+1}=\sup\limits_{[a,b]}\left|f^{(n+1)}(x)\right|$.

2.3-§. Nyuton interpolyatsion ko'phadi

Bizga $[a,b]$ da aniqlangan $f(x)$ funksiyaning $[a,b]$ ga tegishli turli $\{x_k\}_{k=0}^{n}$ nuqtalarda qiymatlari ma'lum bo'lsin.

Quyidagicha aniqlangan

$$f(x_i,x_{i+1})=\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i},\quad i=0,1,\dots,n-1,$$

miqdorlar *birinchi tartibli ayirmalar nisbati* deyiladi, ular yordamida aniqlangan

$$f(x_i,x_{i+1},x_{i+2})=\frac{f(x_{i+1},x_{i+2})-f(x_i,x_{i+1})}{x_{i+2}-x_i},\quad i=0,1,\dots,n-2$$

miqdorlar *ikkinchi tartibli ayirmalar nisbati* deyiladi.

Yuqori tartibli ayirmalar nisbati ham shunday aniqlanadi, masalan, k-tartibli $f(x_i,x_{i+1},\dots,x_{i+k})$ va $f(x_{i+1},x_{i+2},\dots,x_{i+k+1})$ ayirmalar nisbati ma'lum bo'lsa, $(k+1)$-tartibli ayirmalar nisbati

$$f(x_i, x_{i+1}, \dots, x_{i+k+1}) = \frac{f(x_{i+1}, x_{i+2}, \dots, x_{i+k+1}) - f(x_i, x_{i+1}, \dots, x_{i+k})}{x_{i+k+1} - x_i}$$

aniqlanadi, $i = 0, 1, \dots, n-k-1$.

Ayirmalar nisbati quyidagi xossalarga ega.

1-xossa. Algebraik yig'indidan olingan ayirmalar nisbati qo'shiluvchilardan olingan ayirmalar nisbatlarining yig'indisiga teng.

2-xossa. O'zgarmasni ayirmalar nisbati belgisidan tashqariga chiqarish mumkin.

3-xossa. Ayirmalar nisbati o'z argumentlariga nisbatan simmetrik funksiyadir.

4-xossa. m-darajali algebraik ko'phaddan olingan k-tartibli ayirmalar nisbati, agar $k > m$ bo'lsa nolga, $k = m$ da o'zgarmasga va $k < m$ bo'lsa argumentlariga nisbatan $(m-k)$-darajali simmetrik birjinsli ko'phadga teng.

Interpolyatsion ko'phad $L_n(x)$ ning boshqa formasini chiqaramiz. Buning uchun Lagranj interpolyatsion ko'phadi $L_n(x)$ dan x_0 nuqtani ishtirok ettirib, birinchi tartibli ayirmalar nisbatini olamiz:

$$L_n(x, x_0) = \frac{L_n(x) - L_n(x_0)}{x - x_0},$$

bundan

$$L_n(x) = L_n(x_0) + L_n(x, x_0)(x - x_0) \tag{1}$$

hosil bo'ladi.

Endi x_1 nuqtani qo'llab,

$$L_n(x, x_0, x_1) = \frac{L_n(x, x_0) - L_n(x_0, x_1)}{x - x_1}$$

ni hosil qilamiz. Bundan

$$L_n(x, x_0) = L_n(x_0, x_1) + L_n(x, x_0, x_1)(x - x_1) \tag{2}$$

bo'ladi. (2) va (1) dan esa

$$L_n(x)=L_n(x_0)+L_n(x_0,x_1)(x-x_0)+L_n(x,x_0,x_1)(x-x_0)(x-x_1) \qquad (3)$$

hosil bo'ladi.

Mana shu yo'l bilan ketma-ket hamma nuqtalarni qo'llab, $L_n(x)$ dan ayirmalar nisbatini topamiz va natijada quyidagiga ega bo'lamiz:

$$\begin{aligned} L_n(x) = L_n(x_0) &+ L_n(x_0,x_1)(x-x_0) + \\ &+ L_n(x_0,x_1,x_2)(x-x_0)(x-x_1)+...+ \\ &+L_n(x_0,x_1,...,x_n)(x-x_0)(x-x_1)\cdots(x-x_{n-1})+ \\ &+L_n(x,x_0,x_1,...,x_n)(x-x_0)(x-x_1)\cdots(x-x_n) \end{aligned} \qquad (4)$$

$L_n(x)$ ko'phadning darajasi n ga teng bo'lganligi uchun 4-xossaga asosan

$$L_n(x,x_0,x_1,...,x_n)=0$$

bo'ladi va $L_n(x_k)=f(x_k)$, $k=0,1,...,n$ ekanligini e'tiborga olsak (4) ifoda quyidagi ko'rinishga keladi:

$$\begin{aligned} L_n(x)= f(x_0)+f(x_0,x_1)(x-x_0)+f(x_0,x_1,x_2)(x-x_0)(x-x_1)+... \\ +f(x_0,x_1,...,x_n)(x-x_0)(x-x_1)\cdots(x-x_{n-1}). \end{aligned} \qquad (5)$$

Agar $x_0 < x_1 < ... < x_n$ bo'lsa, (5) ifodani *Nyutonning oldga interpolyatsiyalash formulasi* deyiladi.

Agar interpolyatsiyalashga tugun nuqtalarni $x_n > x_{n-1} > ... > x_1 > x_0$ tartibda jalb qilsak, Nyutonning *ortga interpolyatsiyalash* deb ataluvchi

$$\begin{aligned} L_n(x)= f(x_n)+f(x_n,x_{n-1})(x-x_n)+f(x_n,x_{n-1},x_{n-2})(x-x_n)(x-x_{n-1})+... \\ +f(x_n,x_{n-1},...,x_0)(x-x_1)\cdots(x-x_n). \end{aligned} \qquad (6)$$

formulasiga ega bo'lamiz.

Interpolyatsiyalash xatoligi

$$\varepsilon_n(x)=f(x)-L_n(x),$$

ekanligini nazarga olsak, u holda (4) dan

$$\pi_n(x)=f(x,x_0,...,x_n)(x-x_0)(x-x_1)\cdots(x-x_n)$$

ekanligiga ishonch hosil qilamiz. Bundan, o'z navbatida,

$$f(x,x_0,...,x_n)=\frac{f^{(n+1)}(x)}{(n+1)!}$$

hosil bo'ladi.

Interpolyatsion ko'phadning Lagranj hamda Nyuton formasi bilan tanishdik, ular yagona ko'phadning yozilishi jihatdan ikki xil ko'rinishidir.

Nyutonning (5) interpolyatsion ko'phadi $f(x)$ funksiya uchun

$$f(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+...+$$

$$+\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

Teylor formulasining ayirmali analogidir.

Agar tugun nuqtalar berilgan bo'lib, bir nechta funksiyani interpolyatsiyalash lozim bo'lsa, interpolyatsion ko'phadni Lagranj formasini ishlatish, bordi-yu bitta funksiyani interpolyatsiyalashga tugun nuqtalarni ketma-ket jalb qilish talab etilsa, Nyuton interpolyatsion formulasini ishlatish maqsadga muvofiqdir, chunki hisoblash jarayoni ancha kamayadi.

Agar $\{x_k\}_{k=0}^{n}$ – interpolyatsiyalash tugun nuqtalari $x_k=x_0+kh,\ k=0,1,...,n$ ko'rinishda bo'lsa (funksiya jadvalini tuzishda ishlatiladi) va ularni interpolyatsiyalashga jalb etish tartibi berilishiga qarab interpolyatsion ko'phadning turli formalari kelib chiqadi. Bu holda interpolyatsion ko'phad ifodasida boshqa miqdorlar – funksiyaning chekli ayirmalari ishtirok etadi.

Faraz qilaylik, $y=f(x)$ funksiyaning h qadam bilan $x_k=x_0+kh$, nuqtalarda $f(x_k)$ qiymatlari berilgan bo'lsin, $k=0,1,...,n$.

Quyidagicha aniqlangan

$$f_{k+1}-f_k=\Delta f_k,\quad k=0,1,\dots,n-1$$

miqdorlar birinchi tartibli *chekli ayirmalar* deyiladi. Xuddi shuningdek, yuqori tartibli chekli ayirmalar aniqlanadi:

$$\Delta^m f_k=\Delta(\Delta^{m-1}f_k)=\Delta^{m-1}f_{k+1}-\Delta^{m-1}f_k,\ \ k=0,1,\dots,n-m+1.$$

Ayirmalar nisbati bilan chekli ayirmalar orasida quyidagi bog'lanish mavjuddir:

$$f(x_k,x_{k+1},\dots,x_{k+m})=\frac{\Delta^m f_k}{m!h^m},\tag{7}$$

bu yerda $h=x_{k+1}-x_k$, $k=0,1,\dots,n-1$.

Nyutonning oldga (5) va ortga (6) interpolyatsion formulasida qatnashuvchi ayirmalar nisbatini (7) ga asosan chekli ayirmalarga almashtirsak, mos ravishda teng oraliqlar uchun Nyutonning birinchi (oldga) va ikkinchi (ortga) interpolyatsion ko'phadlari hosil bo'ladi. Ularni keltiramiz:

$$\begin{aligned}L_n^{(1)}(x)&=f_0+\frac{\Delta f_0}{1!h}(x-x_0)+\frac{\Delta^2 f_0}{2!h^2}(x-x_0)(x-x_1)+\\&+\frac{\Delta^3 f_0}{3!h^3}(x-x_0)(x-x_1)(x-x_2)+\dots+\frac{\Delta^n f_0}{n!h^n}(x-x_0)(x-x_1)\cdots(x-x_{n-1}),\end{aligned}\tag{8}$$

$$\begin{aligned}L_n^{(2)}(x)&=f_n+\frac{\Delta f_{n-1}}{1!h}(x-x_n)+\frac{\Delta^2 f_{n-2}}{2!h^2}(x-x_n)(x-x_{n-1})+\\&+\frac{\Delta^3 f_{n-3}}{3!h^3}(x-x_n)(x-x_{n-1})(x-x_{n-2})+\dots+\frac{\Delta^n f_0}{n!h^n}(x-x_n)(x-x_{n-1})\cdots(x-x_1).\end{aligned}\tag{9}$$

Interpolyatsiyalashga tugun nuqtalarni quyidagicha

$$\begin{matrix}x_{-n}, & \dots, & x_{-2}, & x_{-1}, & x_0, & x_1, & x_2, & \dots, & x_n\\ 2n+1 & \dots & 5 & 3 & 1 & 2 & 4 & \dots & 2n\end{matrix}\tag{10}$$

yoki

$$\begin{matrix}x_{-n}, & \dots, & x_{-2}, & x_{-1}, & x_0, & x_1, & x_2, & \dots, & x_n\\ 2n & \dots & 4 & 2 & 1 & 3 & 5 & \dots & 2n+1\end{matrix}\tag{11}$$

tartibda jalb etilsa, mos ravishda teng oraliqlar uchun Gaussning birinchi (oldga) va ikkinchi (ortga) interpolyatsion ko'phadlarini hosil qilamiz. Bu ko'phadlarni chiqarishni o'quvchiga havola etamiz.

Interpolyatsiyalashni amalda qo'llaganda uning qoldiq hadini baholashning har doim ham uddasidan chiqmaslik mumkin, chunki qoldiq had ifodasida interpolyatsiyalanuvchi funksiyaning yuqori tartibli hosilasi qatnashishi masalani ancha murakkablashtiradi. Shuning uchun ham yetarlicha ko'p tugunlar olinganda interpolyatsion ko'phadning interpolyatsiyalanuvchi funksiyaga yetarlicha yaxshi yaqinlashishiga ishonch hosil qilish amaliy interpolyatsiyalashda katta ahamiyatga ega. Shu bois ham, interpolyatsiyalash jarayonining yaqinlashishi masalasi tug'iladi. Bu yo'nalishda ko'p tadqiqotlar qilingan, ularni maxsus adabiyotlardan topish mumkin.

2.4-§. Teskari interpolyatsiyalash

Amaliyotda ko'pincha funksiyaning berilgan $\bar{y}$ qiymatiga mos $\bar{x}$ argument qiymatini topish masalasi tez-tez uchrab turadi. Bu masala *teskari interpolyatsiyalash* metodi bilan hal qilinadi. Agar funksiya jadvalining qaralayotgan oralig'ida $y=f(x)$ monoton bo'lsa, u holda bir qiymatli $x=\varphi(y)$ $(f(\varphi(y))=y)$ teskari funksiya mavjud bo'ladi. Bu holda teskari interpolyatsiya $\varphi(y)$ funksiya uchun odatdagi interpolyatsiyalashga keltiriladi. $\bar{x}=\varphi(\bar{y})$ ni topish uchun Lagranj interpolyatsion formulasidan foydalaniladi:

$$\bar{x}=L_n(\bar{y})=\sum_{i=0}^{n} x_i\cdot\frac{\omega_{n+1}(\bar{y})}{(\bar{y}-y_i)\omega'_{n+1}(y_i)}.$$

Buning qoldiq hadi quyidaga teng bo'ladi:

$$\varphi(\bar{y})-L_n(\bar{y})=\frac{\varphi^{(n+1)}(y)}{(n+1)!}\omega_{n+1}(y)$$

Agar $f(x)$ monoton bo'lmasa, u yoki bu interpolyatsion formulani yozib, argumentning ma'lum qiymatlaridan foydalanib va funksiyani ma'lum deb hisoblab, hosil bo'lgan tenglamani biron-bir metod bilan argumentga nisbatan yechiladi.

$$L_n(x) = \overline{y}$$

tenglamani yechib $\overline{x}$ topiladi.

2.5-§. Interpolyatsion ko'phadlarning qoldiq hadi bahosini minimallashtirish masalasi. Chebishev ko'phadlari

Faraz qilaylik, $f(x) \in C_{n+1}[a,b]$ bo'lsin. $[a,b]$ da interpolyatsiyalash tugun nuqtalari $x_0, x_1, \ldots, x_n$ lar qanday tanlansa, $f(x)$ funksiyani interpolyatsiyalashdagi maksimal xatolik eng kichik qiymatga ega bo'ladi, degan savol chiqishi tabiiydir. Bu masala ancha murakkabdir va uni faqat xususiy holdagi $f(x)$ lar uchungina yechish mumkin. Biz ancha sodda masalani yechishni ko'ramiz, ya'ni x_i, $i = 0,1,\ldots,n$ tugun nuqtalar $[a,b]$ da qanday joylashganda $\max\limits_{[a,b]}|\omega_{n+1}(x)|$ miqdor eng kichik bo'ladi, degan savolga javob beramiz. Agar $\max\limits_{[a,b]}|\omega_{n+1}(x)|$ eng kichik qiymatga erishsa, interpolyatsiya xatoligi (6) (2.2-§) ham minimal qiymatga ega bo'ladi.

Soddalik uchun oraliq $[-1,1]$ bo'lsin. Bizga Chebishev ko'phadlari kerak bo'ladi. Bu ko'phadlar $[-1,1]$ oraliqda

$$T_n(x) = \cos(n \arccos x),\ n = 0,1,\ldots \qquad (1)$$

formula bilan aniqlanadi. Xususan, $n = 0, 1$ da

$$T_0(x) = \cos(0 \arccos x) = 1, \qquad (2)$$

$$T_1(x) = \cos(1 \arccos x) = x \qquad (3)$$

ga ega bo'lamiz. Chebishev ko'phadlari uchun

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad (4)$$

rekurrent munosabat o'rinli, bu quyidagi

$$\cos(n+1)\varphi = 2\cos\varphi\cos n\varphi - \cos(n-1)\varphi$$

ayniyatdan $\varphi = \arccos x$ desak kelib chiqadi.

Chebishev ko'phadining xossalari.

1-xossa. Juft (toq) n uchun $T_n(x)$ da x ning faqat juft (toq) darajalari qatnashadi. Bu (2)–(4) formulalardan osongina chiqariladi.

2-xossa. $T_n(x)$ ning bosh koeffitsiyenti $n \geq 1$ da 2^{n-1} ga teng.

Bu ham (2)–(4) formulalardan chiqariladi.

3-xossa. $T_n(x)$ $(-1,1)$ intervalda turli n ta haqiqiy ildizlarga ega, ular quyidagilar:

$$x_i = \cos\frac{(2i+1)\pi}{2n}, \quad i = 0,1,\ldots,n-1.$$

Haqiqatdan, $T_n(x_i) = \cos(n\arccos x_i) = \cos\frac{(2i+1)\pi}{2} = 0, \quad i = 0,1,\ldots,n-1.$

4-xossa. $\max\limits_{[-1,1]}|T_n(x_m)| = 1$ bo'lib,

$$T_n(x_m) = (-1)^m, \qquad (5)$$

bu yerda $x_m = \cos\frac{m\pi}{2n}$, $m = 0,1,\ldots,n.$

Haqiqatdan, (1)ga ko'ra $T_n(x_m) = \cos m\pi = (-1)^m$, $|T_n(x)| \leq 1$, $x \in [-1,1]$. Demak 4-xossa o'rinli.

5-xossa.

$$\bar{T}_n(x) = 2^{1-n}T_n(x), \quad n \geq 1 \qquad (6)$$

ko'phad, bosh koeffitsiyenti birga teng bo'lgan n-darajali ko'phadlar ichida $[-1,1]$ da modulining maksimumi eng kichik bo'lgan yagona ko'phaddir.

Isboti. Teskarisini faraz qilamiz, ya'ni $\bar{P}_n(x) = x^n + a_1x^{n-1} + \cdots + a_n$ ko'phad mavjud bo'lib, u

$$\max_{[-1,1]}|\bar{P}_n(x)| < \max_{[-1,1]}|\bar{T}_n(x)| = 2^{1-n} \tag{7}$$

tengsizlikni qanoatlantirsin. U holda (6) ga asosan, $\bar{T}_n(x)$ ning bosh koeffitsiyenti birga teng bo'lib, $\bar{T}_n(x) - \bar{P}_n(x)$ darajasi $n-1$ ga teng ko'phaddir. (7) ga asosan, $\bar{T}_n(x) - \bar{P}_n(x) \not\equiv 0$. Bundan tashqari, (5)-(7) formulalarga asosan, $\bar{T}_n(x) - \bar{P}_n(x)$ $n+1$ ta $x_m = \cos\frac{m\pi}{2n}$, $m = 0,1,\dots,n$. nuqtalarda noldan farqli va almashuvchi ishorali qiymatlar qabul qiladi. Bu esa, o'z navbatida, darajasi n dan kichik $\bar{T}_n(x) - \bar{P}_n(x)$ ko'phad kami deganda n ta nuqtada nolga teng bo'lishligini anglatadi. Bu ziddiyat 5-xossani isbot etadi.

5-xossaga asosan Chebishevning ko'phadlari $T_n(x)$ *noldan eng kam og'uvchi ko'phad* deyiladi.

$[-1,1]$ oraliqda interpolyatsiyalashning tugun nuqtalar sifatida $T_{n+1}(x)$ ko'phadning ildizlari

$$x_i = \cos\frac{2(i+1)\pi}{2n+2}, \quad i = 0,1,\dots,n \tag{8}$$

larni olaylik. Unda interpolyatsiyalash qoldiq hadida ishtirok etuvchi bosh koeffitsiyenti birga teng bo'lgan $n+1$-darajali ko'phad quyidagicha bo'ladi:

$$\omega_{n+1}(x) = 2^{-n}\,T_{n+1}(x).$$

U holda 4-xossaga asosan interpolyatsiyalashning qoldiq had bahosi

$$\max_{[-1,1]}|f(x) - L_n(x)| \le \frac{M_{n+1}}{(n+1)!\,2^n} \tag{9}$$

ko'rinishga ega bo'ladi, bu yerda $M_{n+1} = \max_{[-1,1]}\left|f^{(n+1)}(x)\right|$.

5-xossaga asosan, (9) bahoni bundan yaxshilab bo'lmaydi. Interpolyatsiyalashning tugun nuqtalari (8) ko'rinishda bo'lganda (9)ni o'ng tomonini kichraytirish mumkin bo'lmagani uchun (8) nuqtalar $[-1,1]$ oraliq uchun optimal tugun nuqtalar bo'ladi.

Agar ixtiyoriy $[a,b]$ oraliq uchun interpolyatsiyalash ko'rilganda

$$x=\frac{1}{2}\left[(b-a)t+b+a\right],\quad t=\frac{1}{b-a}(2x-b-a)$$

almashtirish yordamida $[a,b]$ oraliq $[-1,1]$ ga o'tadi, bu holda $T_{n+1}(t)$ ning ildizlari

$$x_i=\frac{1}{2}\left((b-a)\cos\frac{(2i+1)\pi}{2n+2}+b+a\right),\quad i=0,1,...,n$$

ko'rinishda bo'ladi. Demak

$$\max_{[a,b]}\left|\omega_{n+1}(x)\right|=\frac{(b-a)^{n+1}}{2^{n+1}}\max_{[-1,1]}\left|\overline{T}_{n+1}(t)\right|=\frac{(b-a)^{n+1}}{2^{2n+1}}$$

ga teng, interpolyatsiyalashning xatoligi bahosi esa quyidagicha bo'ladi:

$$\max_{[a,b]}\left|f(x)-L_n(x)\right|\le\frac{M_{n+1}}{(n+1)!}\cdot\frac{(b-a)^{n+1}}{2^{2n+1}}.$$

2.6-§. Oraliqda algebraik ko'phadlar orqali o'rta kvadratik yaqinlashish

Faraz qilaylik, $f(x)$ funksiya $p(x)\ge 0$ vazn bilan $[a,b]$ oraliqda kvadrati bilan integrallanuvchi bo'lsin, ya'ni

$$\int_a^b p(x)f^2(x)\,dx$$

mavjud bo'lsin. Bunday funksiyalarni $L_p^2[a,b]$ fazoga tegishli deyiladi. Bu funksiyani

$$P_n(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x) \qquad (1)$$

umumlashgan ko‘phad bilan o‘rta kvadratik ma’noda yaqinlashtirish masalasini qaraylik, ya’ni $a_0, a_1, \ldots, a_n$ koeffitsiyentlarni shunday topaylikki,

$$\delta_n = \int_a^b p(x)\left[f(x) - P_n(x)\right]^2 dx \qquad (2)$$

ifoda eng kichik qiymat qabul qilsin.

Bu yerda $\{\varphi_k(x)\}_{k=0}^n$ $[a,b]$ da yetarlicha silliq va hisoblash uchun qulay bo‘lgan chiziqli bog‘liqsiz funksiyalar sistemasidir. $\{\varphi_k(x)\}_{k=0}^n$ funksiyalar sistemasi $[a,b]$ da Chebishev sistemasini tashkil etadi, deb hisoblaymiz. δ_n funksiya $a_0, a_1, \ldots, a_n$ larga nisbatan kvadratik ko‘phad va $\delta_n \geq 0$ bo‘lgani uchun uning minimumi mavjud, bu minimumni topish uchun

$$\frac{\partial \delta_n}{\partial a_k} = 0, \quad k = 0, 1, \ldots, n$$

chiziqli algebraik tenglamalar sistemasini yechish kerak bo‘ladi. Bu tenglamalar sistemasi quyidagi ko‘rinishga egadir:

$$\begin{cases} \int_a^b p(x)\left[\sum_{k=0}^n a_k\varphi_k(x) - f(x)\right]\cdot \varphi_0(x)\, dx = 0, \\ \int_a^b p(x)\left[\sum_{k=0}^n a_k\varphi_k(x) - f(x)\right]\cdot \varphi_1(x)\, dx = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \int_a^b p(x)\left[\sum_{k=0}^n a_k\varphi_k(x) - f(x)\right]\cdot \varphi_n(x)\, dx = 0. \end{cases} \qquad (3)$$

Agar $L_p^2[a,b]$ fazodan olingan ixtiyoriy ikki $\varphi(x)$ va $\psi(x)$ funksiya skalyar ko‘paytmasini (φ, ψ) orqali belgilasak:

$$(\varphi, \psi) = \int_a^b p(x)\varphi(x)\psi(x)dx, \tag{4}$$

u holda (3)ni quyidagi ko'rinishda yozish mumkin:

$$\begin{aligned} &a_0(\varphi_0, \varphi_0) + a_1(\varphi_1, \varphi_0) + \cdots + a_n(\varphi_n, \varphi_0) = (f, \varphi_0), \\ &a_0(\varphi_0, \varphi_1) + a_1(\varphi_1, \varphi_1) + \cdots + a_n(\varphi_n, \varphi_1) = (f, \varphi_1), \\ &\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ &a_0(\varphi_0, \varphi_n) + a_1(\varphi_1, \varphi_n) + \cdots + a_n(\varphi_n, \varphi_n) = (f, \varphi_n). \end{aligned} \tag{5}$$

Bu sistema yagona yechimga egadir, chunki uning determinanti Gram determinantidir. Chiziqli bog'liqsiz funksiyalar sistemasidan tuzilgan

$$\Gamma_n = \begin{vmatrix} (\varphi_0, \varphi_0) & (\varphi_1, \varphi_0) & \cdots & (\varphi_n, \varphi_0) \\ (\varphi_0, \varphi_1) & (\varphi_1, \varphi_1) & \cdots & (\varphi_n, \varphi_1) \\ \ldots & \ldots & \ldots & \ldots \\ (\varphi_0, \varphi_n) & (\varphi_1, \varphi_n) & \cdots & (\varphi_n, \varphi_n) \end{vmatrix}.$$

determinant *Gram determinanti* deyiladi va uni noldan farqliligini ko'rsatamiz. Faraz qilaylik, aksincha, ya'ni $\Gamma_n = 0$ bo'lsin. U holda (5) sistemaga mos keladigan bir jinsli chiziqli algebraik tenglamalar sistemasi

$$a_0(\varphi_0, \varphi_i) + a_1(\varphi_1, \varphi_i) + \cdots + a_n(\varphi_n, \varphi_i) = 0, \quad i = 0,1,\ldots,n \tag{6}$$

kamida bitta trivial bo'lmagan yechimga ega bo'lishi kerak, ya'ni shunday $a_0, a_1, \ldots, a_n$ sonlar topilishi kerakki, ularning kamida bittasi noldan farqli bo'lib, (5) sistemani qanoatlantirsin. (6) sistemaning tenglamalarini mos ravishda $a_0, a_1, \ldots, a_n$ larga ko'paytirib yig'amiz va (4) ni e'tiborga olsak, quyidagi hosil bo'ladi:

$$\int_a^b p(x)\left[a_0\varphi_0(x) + a_1\varphi_1(x) + \cdots + a_n\varphi_n(x)\right]^2 dx = 0.$$

Bunday bo'lishi mumkin emas, chunki $\{\varphi_k(x)\}_{k=0}^{n}$ chiziqli bog'liqsiz funksiyalar sistemasi bo'lib, $a_0, a_1, \dots, a_n$ larning kamida bittasi noldan farqliligi sababli $\left[a_0\varphi_0(x) + a_1\varphi_1(x) + \dots + a_n\varphi_n(x)\right]^2$ ko'phad aynan nolga teng emas. Demak, Gram determinanti noldan farqli va (5) sistema yagona yechimga ega.

Agar $[a,b]$ oraliqda $p(x) \geq 0$ vazn funksiya bilan $\{\varphi_k(x)\}_{k=0}^{n}$ funksiyalar sistemasi ortogonal ko'phadlar sistemasini, ya'ni

$$\left(\varphi_k(x), \varphi_l(x)\right) = 0, \ k \neq l$$

tashkil etsa, u holda (5) tenglamalar sistemasining har bir tenglamasi bitta noma'lumga bog'liq bo'lib, a_k koeffitsiyentlar quyidagicha aniqlanadi:

$$a_k = \frac{(f, \varphi_k)}{(\varphi_k, \varphi_k)}, \quad k = 0, 1, \dots, n.$$

Agar $\{\varphi_k(x)\}_{k=0}^{n}$ $[a,b]$ da $p(x) \geq 0$ vazn funksiya bilan ortonormal ko'phadlar sistemasini tashkil etsa, u holda a_k koeffitsiyentlar quyidagicha aniqlanadi:

$$a_k = (f, \varphi_k) = \int_a^b p(x)\, f(x)\, \varphi_k(x)\, dx, \ \ k = 0, 1, \dots, n.$$

Bu holda eng kichik og'ish

$$\delta_n^2 = \int_a^b p(x)\left[\sum_{i=0}^{n} a_i\varphi_i(x) - f(x)\right]^2 dx =$$

$$\int_a^b p(x) f^2(x) dx - 2\sum_{i=0}^{n} a_i \int_a^b p(x) f(x) \varphi_i dx + \sum_{i=0}^{n}\sum_{k=0}^{n} a_i a_k \int_a^b p(x)\varphi_i(x)\varphi_k(x) dx =$$

$$\int_a^b p(x) f^2(x) dx - 2\sum_{i=0}^{n} a_i^2 + \sum_{i=0}^{n} a_i^2,$$

ya'ni

$$\delta_n^2 = \int_a^b p(x) f^2(x)dx - \sum_{k=0}^{n} a_k^2$$

bilan xarakterlanadi.

Quyida hisoblash matematikasida ko'p qo'llaniladigan ortogonal ko'phadlar sistemalarini keltiramiz [11].

Yakobi ko'phadlari. Quyidagi

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]$$

ko'phadlar *Yakobi ko'phadlari* deyiladi. Ular $[-1,1]$ oraliqda

$$p(x) = (1-x)^{\alpha}(1+x)^{\beta}$$

vazn funksiya bilan ortogonal ko'phadlar sistemasini tashkil etadi. Ularning normalari:

$$\left\|P_n^{(\alpha,\beta)}\right\| = \sqrt{\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left[P_n^{(\alpha,\beta)}(x)\right]^2 dx} =$$

$$= \left[\frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}\right]^{1/2}.$$

Ular uchun quyidagi rekurrent formula o'rinli:

$$(\alpha+\beta+2n)(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)xP_n^{(\alpha,\beta)}(x) =$$

$$= 2(n+1)(\alpha+\beta+n+1)(\alpha+\beta+2n)P_{n+1}^{(\alpha,\beta)}(x) +$$

$$+(\beta^2-\alpha^2)(\alpha+\beta+2n+1)P_{n+1}^{(\alpha,\beta)}(x) +$$

$$+ 2(\alpha+n)(\beta+n)(\alpha+\beta+2n+2)P_{n-1}^{(\alpha,\beta)}(x).$$

Lejandr ko'phadlari. Yakobi ko'phadlarining $\alpha=\beta=0$ va $p(x)=1$ bo'lgandagi xususiy holi *Lejandr ko'phadlari* deb yuritiladi va ular Rodriga formulasi

$$L_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

bilan aniqlanadi. Ularning normalari

$$\|L_n\| = \sqrt{\int_{-1}^{1} L_n^2(x)dx} = \sqrt{\frac{2}{2n+1}}$$

bo'lib, rekurrent munosabat esa

$$(n+1)L_{n+1}(x) = (2n+1)\cdot x\cdot L_n(x) - n L_{n-1}(x)$$

dan iborat.

Chebishevning birinchi tur ko'phadlari. Ular quyidagicha

$$T_n(x) = \cos(n \arccos x), \quad |x| \le 1,$$

aniqlanib $[-1,1]$ oraliqda $p(x) = (1-x^2)^{-\frac{1}{2}}$ vazn bilan ortogonal ko'phadlar sistemasini tashkil etadi. Bu ko'phadning normasi

$$\|T_n\| = \sqrt{\int_{-1}^{1} \frac{T_n^2(x)}{\sqrt{1-x^2}}dx} = \begin{cases} \sqrt{\pi}, & \text{agar } n = 0 \text{ bo'lsa}, \\ \sqrt{\frac{\pi}{2}}, & \text{agar } n > 0 \text{ bo'lsa}, \end{cases}$$

ga teng. Rekurrent munosabat esa

$$T_{n+1}(x) = 2\cdot x\cdot T_n(x) - T_{n-1}(x)$$

formula bilan aniqlanadi.

$[-1,1]$ da $p(x) = (1-x^2)^{-\frac{1}{2}}$ vaznda kvadrati bilan integrallanuvchi $f(x)$ funksiya uchun Chebishevning birinchi tur ko'phadlari yordamida topilishi lozim bo'lgan eng yaxshi yaqinlashuvchi $\sum_{k=0}^{n} a_k T_k(x)$ ko'phadning koeffitsiyentlari quyidagi formulalar bilan aniqlanadi:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(\cos\theta)d\theta, \quad a_k = \frac{2}{\pi}\int_0^{\pi} f(\cos\theta)\cos k\theta\, d\theta \quad (k > 1).$$

Eng kichik og'ish esa

$$\delta_n^2 = \int_{-1}^{1}\left[f(x) - \sum_{k=0}^{n} a_k T_k(x)\right]^2 \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^{1}\frac{f^2(x)}{\sqrt{1-x^2}}dx - \frac{\pi}{2}\left[2a_0^2 + a_1^2 + \dots + a_n^2\right]$$

formula bilan ifodalanadi.

Chebishevning ikkinchi tur ko'phadlari. Bu ko'phad $[-1,1]$ oraliqda $p(x)=\sqrt{1-x^2}$ vazn bilan ortogonal ko'phadlar sistemasini tashkil etadi va u

$$u_n(x)=\frac{\sin(n+1)\arccos x}{\sqrt{1-x^2}}=\frac{T'_{n+1}(x)}{n+1},\quad n=0,1,2,...$$

formula bilan aniqlanadi. Uning normasi

$$\|u_n\|=\sqrt{\int_{-1}^{1}\sqrt{1-x^2}\;u_n^2(x)dx}=\sqrt{\frac{\pi}{2}}$$

ga teng bo'lib, rekurrent munosabat

$$u_{n+1}(x)=2xu_n(x)-u_{n-1}(x)$$

dan iboratdir.

$[-1,1]$ oraliqda $p(x)=\sqrt{1-x^2}$ vaznda kvadrati bilan integrallanuvchi $f(x)$ funksiya uchun Chebishevning ikkinchi tur ko'phadlari yordamida tuzilgan eng yaxshi yaqinlashuvchi ko'phadning koeffitsiyentlari

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\cos\theta)\sin\theta\sin(k+1)\theta\,d\theta,\quad k=0,1,...,n$$

formula bilan hisoblanib, eng kichik og'ish miqdori esa

$$\delta_n^{\,2}=\int_{-1}^{1}\left[f(x)-\sum_{k=0}^{n}a_k u_k(x)\right]^2\sqrt{1-x^2}dx=\int_{-1}^{1}\sqrt{1-x^2}\;f^2(x)dx-\frac{\pi}{2}\sum_{k=0}^{n}a_k^2$$

formula bilan aniqlanadi.

Lagerr ko'phadlari. Bu ko'phad $[0,\infty)$ oraliqda, $p(x)=e^{-x}x^{\alpha}$, $\alpha>-1$ vazn bilan ortogonal bo'lgan ko'phad

$$L_n^{(\alpha)}(x)=(-1)^n x^{-\alpha}e^{x}\frac{d^n}{dx^n}(x^{\alpha+n}e^{-x})$$

formula bilan aniqlanadi. Buning normasi

$$\left\|L_n^{(\alpha)}\right\| = \sqrt{\int_0^\infty x^\alpha e^{-x}\left[L_n^{(\alpha)}(x)\right]^2 dx} = \sqrt{n!\,\Gamma(\alpha+n+1)}$$

ga teng bo'lib, ular uchun

$$L_{n+1}^{(\alpha)}(x) - (x-\alpha-2n-1)L_n^{(\alpha)}(x) + n(\alpha+n)L_{n-1}^{(\alpha)}(x) = 0$$

rekurrent munosabat o'rinlidir.

Agar $f(x)\in[0,\infty)$ bo'lib, $p(x)=e^{-x}x^\alpha$ vaznda kvadrati bilan integrallanuvchi bo'lsa, u funksiyaga o'rta kvadratik ma'noda eng yaxshi yaqinlashuvchi ko'phadning koeffitsiyentlari

$$a_k = \frac{1}{k!\,\Gamma(\alpha+k+1)}\int_0^\infty x^\alpha e^{-x} f(x) L_n^{(\alpha)}(x)dx, \quad k=0,1,\dots,n$$

formula bilan aniqlanadi.

Ermit ko'phadlari. Bu ko'phadlar $(-\infty,+\infty)$ oraliqda $p(x)=e^{-x^2}$ vazn bilan ortogonal bo'lgan ko'phadlar sistemasini tashkil etib

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$$

formula bilan aniqlanadi. Uning normasi

$$\left\|H_n\right\| = \sqrt{\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)dx} = \sqrt{2^n n!\sqrt{\pi}}$$

ga teng bo'lib, uning uchun

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

rekurrent munosabat o'rinli.

Agar $f(x)$ funksiya $(-\infty,+\infty)$ oraliqda $p(x)=e^{-x^2}$ vaznda kvadrati bilan integrallanuvchi bo'lsa, unga o'rta kvadratik ma'noda eng yaxshi yaqinlashuvchi ko'phadning koeffitsiyentlari

$$a_k = \frac{(-1)^k}{2^k k!\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x^2} f(x) H_k(x)dx$$

formula bilan aniqlanadi.

Ortogonal ko'phadlar haqida to'liqroq ma'lumotlarni [20], [21] dan topish mumkin.

2.7-§. Jadval bilan berilgan funksiyalarni o'rta kvadratik ma'noda yaqinlashtirish

Bizga $y = f(x)$ funksiyaning $[a,b]$ ga tegishli turli $x_0, x_1, \ldots, x_n$ nuqtalarda qiymatlari berilgan bo'lsin. Bu funksiyani u yoki bu maqsad uchun interpolyatsiyalashdan boshqa qulay va qandaydir ma'noda aniq analitik ko'rinishini $[a,b]$ oraliqda topish kerak bo'lsin. Interpolyatsiyalash apparatidan foydalanish quyidagi ikki sababga ko'ra maqsadga muvofiq emas. Birinchidan, agar tugun nuqtalar soni katta bo'lsa, u holda interpolyatsion ko'phadning ifodasi qo'pollashadi (juda ko'p hisoblashlar o'tkaziladi). Ikkinchidan, jadvaldagi qiymatlarda tasodifiy xatoliklar mavjud bo'lsa, bu xatoliklar interpolyatsion ko'phadda to'g'ridan-to'g'ri ishtirok etib, funksiyaning haqiqiy o'zgarish holatini akslantirmaydi.

Bizga $[a,b]$ da aniqlangan va chiziqli bog'liqsiz $\varphi_0(x), \varphi_1(x), \ldots, \varphi_m(x)$ $m \le n$ funksiyalar sistemasi berilgan bo'lsin. Ular orqali

$$P_m(x) = \sum_{i=0}^{m} a_i \varphi_i(x) \tag{1}$$

umumlashgan ko'phadni quraylikki,

$$\Delta^2 = \sum_{k=0}^{n} \left[f(x_k) - P_m(x_k) \right]^2 \tag{2}$$

ifoda eng kichik qiymat qabul qilsin. Agar $f(x_k)$ larning aniqligi bir xil emasligi ma'lum bo'lsa, vazn deb ataluvchi $p_k > 0$, $\sum_{k=0}^{n} p_k = 1$ larni kiritib,

$$\Delta^2 = \sum_{k=0}^{n} p_k\left[f(x_k) - \mathrm{P}_m(x_k)\right]^2 \tag{3}$$

ning minimumini topish kerak bo'ladi.

(3) $a_0, a_1, \dots, a_m$ larga nisbatan kvadratik ko'phad, undan a_k, $k = 0,1,\dots,m$ larga nisbatan xususiy hosilalar olib nolga tenglaymiz.

$$\frac{\partial\Delta}{\partial a_k} = 2\sum_{i=0}^{n} p_i\left[f(x_i) - \mathrm{P}_m(x_i)\right]\cdot\varphi_k(x_i) = 0, \quad k = 0,1,\dots,m$$

va $S_{jk} = \sum\limits_{i=0}^{n} p_i\varphi_j(x_i)\varphi_k(x_i)$; $\beta_k = \sum\limits_{i=0}^{n} p_i f(x_i)\varphi_k(x_i)$ belgilashlarni kiritib, a_k larga nisbatan

$$\begin{cases} a_0S_{00} + a_1S_{10} + \dots + a_mS_{m0} = \beta_0 \\ a_0S_{01} + a_1S_{11} + \dots + a_mS_{m1} = \beta_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_0S_{0m} + a_1S_{1m} + \dots + a_mS_{mm} = \beta_m \end{cases} \tag{4}$$

chiziqli algebraik tenglamalar sistemasini hosil qilamiz. Bu tenglamalar sistemasining determinanti Gram determinantidir. U noldan farqli, chunki $\{\varphi_k(x)\}_{k=0}^{m}$ funksiyalar sistemasi chiziqli bog'liqsiz va x_i, $i = 0,1,\dots,n$ tugun nuqtalar turli. Demak, (4) yagona yechimga ega.

2.8-§. Kubik splayn bilan yaqinlashish

Quyidagi to'rt shartni qanoatlantiruvchi ushbu $S_3(x)$ funksiya interpolyatsion kubik splayn deyiladi:

1. Har bir $[x_i, x_{i+1}]$ $\left(i = \overline{0, n-1}\right)$ oraliqda $S_3(x) \in H_3(P)$ – darajasi uchdan oshmaydigan ko'phadlar to'plami.

2. $S_3(x) \in C_2[a,b]$.

3. $S_3(x_i)=f(x_i)$ $\left(i=\overline{0,n-1}\right)$.

4. $S_3''(x)$ uchun

$$S_3''(a)=S_3''(b)=0 \tag{1}$$

chegaraviy shartlar o'rinli.

$S_3''(x)$ $[x_{i-1},x_i]$ kesmada uzluksiz bo'lganligidan $x_{i-1}\le x\le x_i$ da ushbu

$$S_3''(x)=M_{i-1}\frac{x_i-x}{h_i}+M_i\frac{x-x_{i-1}}{h_i} \tag{2}$$

tenglikni yozish mumkin. Bu yerda $h_i=x_i-x_{i-1}$, $M_i=S_3''(x_i)$.

(2) tenglikni ikki marta integrallaymiz:

$$S_3(x)=M_{i-1}\frac{(x_i-x)^3}{6h_i}+M_i\frac{(x-x_{i-1})^3}{6h_i}+A_i\frac{(x_i-x)}{h_i}+B_i\frac{(x-x_{i-1})}{h_i}, \tag{3}$$

bunda A_i va B_i integrallash doimiylari bo'lib, ular ta'rifning uchinchi shartidan topiladi, ya'ni (3) da $x=x_{i-1}$, $x=x_i$ deb, mos ravishda

$$M_{i-1}\frac{h_i^2}{6}+A_i=f_{i-1},\quad M_i\frac{h_i^2}{6}+B_i=f_i$$

larni hosil qilamiz. Bundan A_i va B_i ni topib (3)ga qo'yib, quyidagiga ega bo'lish mumkin:

$$S_3(x)=M_{i-1}\frac{(x_i-x)^3}{6h_i}+M_i\frac{(x-x_{i-1})^3}{6h_i}+\left(f_{i-1}-\frac{M_{i-1}h_i^2}{6}\right)\frac{(x_i-x)}{h_i}+$$

$$+\left(f_i-\frac{M_ih_i^2}{6}\right)\frac{(x-x_{i-1})}{h_i}. \tag{4}$$

(4) dan hosila olamiz:

$$S_3'(x)=-M_{i-1}\frac{(x_i-x)^2}{2h_i}+M_i\frac{(x-x_{i-1})^2}{2h_i}+\frac{f_i-f_{i-1}}{h_i}-\frac{M_i-M_{i-1}}{6}h_i \tag{5}$$

(5) ni $[x_i, x_{i+1}]$ kesma uchun yozamiz:

$$S_3'(x) = -M_i \frac{(x_{i+1}-x)^2}{2h_{i+1}} + M_{i+1}\frac{(x-x_i)^2}{2h_{i+1}} + \frac{f_{i+1}-f_i}{h_{i+1}} - \frac{M_{i+1}-M_i}{6}h_{i+1} \qquad (6)$$

Ta'rifning ikkinchi shartiga ko'ra $S_3'(x)$ va $S_3''(x)$ funksiyalar $[a,b]$ da uzluksiz. $S_3'(x)$ ning $x_1, x_2, \dots, x_{n-1}$ nuqtalarda uzluksizligidan foydalanib, (5) va (6) tengliklardan quyidagi $n-1$ ta tenglamaga ega bo'lamiz:

$$\frac{h_i}{6}M_{i-1} + \frac{h_i+h_{i+1}}{3}M_i + \frac{h_{i+1}}{6}M_{i+1} = \frac{f_{i+1}-f_i}{h_{i+1}} - \frac{f_i-f_{i-1}}{h_i}. \qquad (7)$$

Bu tenglamalarni (1) bilan to'ldirib hamda

$$a_i = \frac{h_i}{6},\; b_i = \frac{h_i+h_{i+1}}{3},\; c_i = \frac{h_{i+1}}{6},\; d_i = \frac{f_{i+1}-f_i}{h_{i+1}} - \frac{f_i-f_{i-1}}{h_i} \qquad (8)$$

belgilashlarni kiritsak, u holda $M_1, M_2, \dots, M_{n-1}$ noma'lumlarni topish uchun

$$\left.\begin{aligned} b_1M_1 + c_1M_2 &= d_1 \\ a_2M_1 + b_2M_2 + c_2M_3 &= d_2 \\ a_3M_2 + b_3M_3 + c_3M_4 &= d_3 \\ \dots\;\dots\;\dots\;\dots\;&\dots\;\dots \\ a_{n-2}M_{n-3} + b_{n-2}M_{n-2} + c_{n-2}M_{n-1} &= d_{n-2} \\ a_{n-1}M_{n-2} + b_{n-1}M_{n-1} &= d_{n-1} \end{aligned}\right\} \qquad (9)$$

tenglamalar sistemasini hosil qilamiz. Bu tenglamalar sistemasining matritsasi (8) ga ko'ra salmoqli bosh diagonalga ega bo'lganligidan ixtiyoriy f_i $(i=\overline{0,n})$ uchun yagona yechimga ega [1].

(9) tenglamalar sistemasi haydash usuli bilan yechiladi. Uni quyida keltiramiz:

$$p_k = a_k q_{k-1} + b_k \quad (q_0 = 0),$$

$$q_k = -\frac{c_k}{p_k},\; u_k = \frac{d_k - a_k u_{k-1}}{p_k},\; u_0 = 0 \;\left(k = \overline{1,n-1}\right)$$

yordamchi miqdorlarni hosil qilamiz. So'ng

$$M_k = q_k M_{k+1} + u_k \quad \left(k = \overline{n-2,1}\right),$$
$$M_{n-1} = u_{n-1}$$

dan $M_{n-1}, M_{n-2}, \ldots, M_1$ larni aniqlaymiz.

Bobga tegishli tayanch so'zlar: interpolyatsion ko'phad, interpolyatsiya xatoligi, ayirmalar nisbati, chekli ayirmalar, interpolyatsiya xatoligi, teskari interpolyatsiya, o'rta kvadratik yaqinlashish, splayn, haydash usuli.

Savollar va topshiriqlar

1. Algebraik interpolyatsiyalash masalasining qo'yilishi va yechimning yagonaligini tushuntiring.

2. Interpolyatsion ko'phadning Lagranj formasini yozing.

3. Interpolyatsiyalashning fundamental ko'phadlarini yozing.

4. Interpolyatsiyalashning qoldiq hadini yozing.

5. Ayirmalar nisbati va ularning xossalari.

6. Interpolyatsiyalashga tugun nuqtalarni jalb etish tartibiga ko'ra qanday interpolyatsion ko'phadlar hosil bo'ladi?

7. Chekli ayirmalar va ularning xossalari.

8. Ayirmalar nisbati bilan chekli ayirmalar orasidagi bog'lanish.

9. Teng oraliqlar uchun Nyutonning birinchi (oldga) va ikkinchi (ortga) interpolyatsion ko'phadlarini tuzing.

10. Ayirmalar nisbati bilan hosila orasidagi bog'lanish qanday?

11. Gaussning birinchi va ikkinchi interpolyatsion ko'phadlarini yozing.

12. Teskari interpolyatsion masalani hal etishni tushuntiring.

13. $T_n(x) = \cos(n \arccos x), n = 0,1,\ldots$ Chebishev ko'phadlari uchun rekurrent formula chiqaring.

14. Chebishev ko'phadlarining xossalari.

15. Oraliqda algebraik ko'phad orqali o'rta kvadratik yaqinlashish masalasining qo'yilishi.

16. Gram determinantining nolga tengmasligini tushuntiring.

17. Chebishevning birinchi va ikkinchi tur ko'phadlarining normasini hisoblang.

18. Lejandr ko'phadlarining normasini hisoblang.

19. Agar $\{\varphi_k(x)\}_{k=0}^{n}$ $[a,b]$ da $p(x)\geq 0$ vazn funksiya bilan ortonormal ko'phadlar sistemasini tashkil etsa, o'rtacha kvadratik ma'noda yaqinlashuvchi ko'phadning koeffitsiyentlari

$$a_k = (f,\varphi_k) = \int_a^b p(x)f(x)\varphi_k(x)dx, \quad k = 0,1,...,n$$

formula bilan eng kichik og'ish miqdori esa

$$\delta_n^2 = \int_a^b p(x)f^2(x)dx - \sum_{i=0}^{n} a_i^2$$

formula bilan aniqlanishini ko'rsating.

20. Kubik splayn ta'rifini ayting.

Misol 1. $f(x) = \sqrt{x}$ funksiyani $x_0 = 100$, $x_1 = 121$, $x_2 = 144$ nuqtalarni ishlatib, ikkinchi tartibli Lagranj interpolyatsion ko'phad bilan approksimatsiya eting va xatolikni $x = 116$ bo'lganda baholang.

Yechish.

$$\omega_3(x) = (x-100)(x-121)(x-144),$$

$$\omega_3'(x) = (x-121)(x-144) + (x-100)(x-144) + (x-100)(x-121),$$

$$\omega_3'(100) = 11\cdot 44, \ \omega_3'(121) = -21\cdot 23, \ \omega_3'(144) = 44 * 23.$$

$$L_2(x) = \frac{(x-121)(x-144)}{11\cdot 44}\cdot 10 + \frac{(x-100)(x-144)}{-21\cdot 23}\cdot 11 + \frac{(x-100)(x-121)}{44\cdot 23}\cdot 12,$$

$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}},\ f''(x)=-\frac{1}{4}x^{-\frac{3}{2}},\ f'''(x)=\frac{3}{8}x^{-\frac{5}{2}},$$

$$\max_{[100,144]}|f'''(x)|=\frac{3}{8}\cdot 100^{-\frac{5}{2}}=\frac{3}{8}\cdot 10^{-5}$$

$$\left|\sqrt{116}-L_2(116)\right|\le\frac{3}{8}10^{-5}\cdot\frac{1}{3!}\left|(116-100)(116-121)(116-144)\right|=$$
$$=\frac{1}{16}10^{-5}\cdot 16\cdot 5\cdot 28=1{,}4\cdot 10^{-3}.$$

Bu $\left|f(x)-L_2(x)\right|\le\frac{\max\limits_{[100,144]}|f'''(x)|}{3!}|\omega_3(x)|$ bahodan foydalanib topildi.

Agar [100,144] oraliqda xatolikning eng katta qiymatini ko‘rsatish lozim bo‘lsa

$$\max_{[100,144]}|f(x)-L_2(x)|\le\frac{\max\limits_{[100,144]}|f'''(x)|}{3!}\max_{[100,144]}|\omega_3(x)|$$

formula bilan baholanadi yani

$$\max_{[100,144]}|f(x)-L_2(x)|\le\frac{3}{8}10^{-5}\frac{1}{3}\max_{[100,144]}|\omega_3(x)|\approx 2{,}5\cdot 10^{-3}$$

Misol 2. $y=\cos x$ funksiyaning jadvali $h=5^\circ$ qadam bilan $\left[10^\circ,35^\circ\right]$ oraliqda berilgan:

x	10°	15°	20°	25°	30°	35°
$\cos x$	0,9848	0,9659	0,9397	0,9063	0,8660	0,8192

$\cos 11^\circ$ va $\cos 36^\circ$ ni taqriban toping.

Yechish.

Quyidagi chekli ayirmalar jadvalini tuzamiz:

x	$\cos x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10°	0,9848					
		−0,0189				
15°	0,9659		−0,0073			
		-0,0262		0,0001		
20°	0,9397		-0,0072		0,0002	
		-0,0334		0,0003		-0,0001
25°	0,9063		-0,0069		0,0001	
		-0,0403		0,0004		
30°	0,8660		−0,0064			
		−0,0468				
35°	0,8192					

$\cos 11°$ ni hisoblash uchun $x_0 = 10°$ va $x = 11°$ deb olamiz. Yuqoridagi jadvalda $\Delta^4 y$ larni taxminan o'zgarmas deb olamiz. $q = \frac{x - x_0}{h} = \frac{11° - 10°}{5°} = 0,2$. Nyutonning birinchi interpolyatsion formulasiga ko'ra jadvaldagi tagiga chiziqcha tortilgan chekli ayirmalarni ishlatib quyidagiga ega bo'lamiz.

$$\cos 11° \cong 0,9848 + 0,2 \cdot (-0,0189) + \frac{0,2 \cdot (0,8)}{2!} \cdot (-0,0073) +$$

$$+ \frac{0,2 \cdot (-0,8) \cdot (-1,8)}{3!} \cdot 0,0001 = 0,9816$$

Eslatib o'tamiz, to'rt xonali matematik jadvaldagi qiymat ham shunday.

$\cos 36°$ ni hisoblash uchun $x_n = 35°$ va $x = 36°$ deb olamiz.

Unda $q = \frac{x - x_n}{h} = \frac{36° - 35°}{5°} = 0,2$ bo'ladi. Nyutonning ikkinchi interpolyatsion formulasida jadvalning quyi og'masida joylashgan (tagiga ikki chiziq tortilgan) chekli ayirmalar ishtirok etadi.

$$\cos 36^\circ \cong 0,8192 + 0,2\cdot(-0,0468) + \frac{0,2\cdot 1,2}{2!}\cdot(-0,0064) +$$

$$+\frac{0,2\cdot 1,2\cdot 1,8}{3!}\cdot 0,0004 = 0,8091$$

To‘rt xonali matematik jadvalda esa $\cos 36^\circ = 0,8090$.

Misol 3. Jadval bilan berilgan funksiyaning Lagranj ko‘phadi bilan yaqinlashtiring:

x	0	1	2	5
$f(x)$	2	3	12	147

Yechish.

$$P_3(x) = \sum_{i=0}^{3} \frac{\omega_4(x)}{(x-x_i)\omega_4'(x_i)} f(x_i) = \frac{(x-1)(x-2)(x-5)}{(-1)(-2)(-5)}\cdot 0 +$$

$$\frac{x(x-2)(x-5)}{1\cdot(-1)(-4)}\cdot 3 + \frac{x(x-1)(x-5)}{2\cdot 1\cdot(-3)}\cdot 12 + \frac{x(x-1)(x-2)}{5\cdot 4 3}\cdot 147 = x^3 + x^2 - x + 2.$$

Misol 4. $\sin x$ ning $x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ dagi qiymatlari berilgan bo‘lsa, $\sin\frac{\pi}{12}$ ni hisoblashdagi xatolikni baholang.

Yechish. $n = 4$ bo‘lib, xatolik bahosi

$$R \le \frac{M_5}{5!}\max\left|x\left(x-\frac{\pi}{6}\right)\left(x-\frac{\pi}{4}\right)\left(x-\frac{\pi}{3}\right)\left(x-\frac{\pi}{2}\right)\right|$$ bo‘ladi, bu yerda

$$M_5 = \max_{\left[0,\frac{\pi}{2}\right]}\left|(\sin x)^{(V)}\right| = \max_{\left[0,\frac{\pi}{2}\right]}\left|-\cos x\right| \le 1$$

$$R_4 \le \frac{1}{5!}\cdot\frac{\pi}{12}\cdot\frac{\pi}{12}\cdot\frac{\pi}{4}\cdot\frac{\pi}{6}\cdot\frac{5\pi}{12} = \frac{1}{4}\cdot\frac{\pi^5}{12^5} \le \frac{10\cdot 10\cdot 3}{4\cdot 12^5} = \frac{300}{4\cdot 12^5} = \frac{75}{12^5} = \frac{25}{4\cdot 12^4} =$$

$$= \frac{6,25}{12^4} = \frac{6,25}{20736} = 0,0003 = 0,3\cdot 10^{-3}.$$

Misol 5. $S(n) = 1^2 + 2^2 + ... + n^2$ ni hisoblash uchun formula chiqaring.

Yechish. Agar $S(n)$ n ga nisbatan k-tartibli algebraik ko‘phad bo‘lsa, $(k+1)$-tartibli chekli ayirmalar nolga teng bo‘ladi. Chekli ayirmalar jadvalini tuzamiz.

n	$S(n)$	ΔS	$\Delta^2 S$	$\Delta^3 S$	$\Delta^4 S$
1	1				
		4			
2	5		5		
		9		2	
3	14		7		0
		16		2	
4	30		9		
		25			
5	55				

Jadvaldan ko‘rinib turibdiki, $S(n) = P_3(n)$ ekan, ya’ni

$$S(n) = P_3(n) = 1 + 4\frac{n-1}{1!} + \frac{5(n-1)(n-2)}{2!} + \frac{2(n-1)(n-2)(n-3)}{3!} =$$

$$= \frac{6+24n-24+15n^2-45n+30+2n^3-12n^2+22n-12}{6} =$$

$$= \frac{2n^3+3n^2+n}{6} = \frac{n(n+1)(2n+1)}{6}.$$

Misol 6. $h = 0{,}001$ qadam bilan $[1;10]$ oraliqdagi sonlarning natural logarifmlari berilgan. Chiziqli interpolyatsiyalashning xatoligini baholang.

Yechish. Nyutonning birinchi interpolyatsion formulasining xatoligidan foydalanamiz:

$$R_1(x) = \frac{(\ln x)''}{2!}(x - x_i)(x - x_{i+1}),$$

$$|R_1(x)| \le \frac{1}{2}\max\left|-\frac{1}{x^2}\right|\left|(x - x_i)(x - x_{i+1})\right| \le \frac{1}{2}\cdot 1\cdot\frac{h^2}{4} = \frac{h^2}{8},$$

$$|R_1(x)| = 0{,}125\cdot 10^{-6}.$$

Misol 7. $f(x)=\sqrt{x}$ funksiyani $p(x)=1-x$ vazn funksiya bilan $[0;1]$ oraliqda birinchi tartibli algebraik ko'phad bilan o'rta kvadratik ma'noda yaqinlashtiring.

Yechish. $P_1(x)=a_0+a_1x$ – qandaydir ko'phad.

Bu yerda $\varphi_0(x)=1$, $\varphi_1(x)=x$. Noma'lum a_0, a_1 –koeffitsiyentlar

$$\begin{cases}(\varphi_0,\varphi_0)a_0+(\varphi_1,\varphi_0)a_1=(f,\varphi_0)\\(\varphi_0,\varphi_1)a_0+(\varphi_1,\varphi_1)a_1=(f,\varphi_1)\end{cases},$$

tenglamalar sistemasidan aniqlanadi:

$$(\varphi_0,\varphi_0)=\int_0^1(1-x)dx=\left(x-\frac{x^2}{2}\right)\Big|_0^1=\frac{1}{2},$$

$$(\varphi_0,\varphi_1)=(\varphi_1,\varphi_0)=\int_0^1(1-x)xdx=\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^1=\frac{1}{6},$$

$$(\varphi_1,\varphi_1)=\int_0^1(1-x)x^2dx=\left(\frac{x^3}{3}-\frac{x^4}{4}\right)\Big|_0^1=\frac{1}{12},$$

$$(f,\varphi_0)=\int_0^1(1-x)\sqrt{x}dx=\frac{4}{15},\quad (f,\varphi_1)=\int_0^1(1-x)\sqrt{x}xdx=\frac{4}{35}.$$

Demak, yuqoridagi tenglamalar sistemasi

$$\begin{cases}\frac{1}{2}a_0+\frac{1}{6}a_1=\frac{4}{15}\\\frac{1}{6}a_0+\frac{1}{12}a_1=\frac{4}{35}\end{cases}$$

ko'rinishga ega bo'lib, uning yechimlari

$$a_0=\frac{8}{35},\ a_1=\frac{32}{35}$$

bo'ladi.

$$P_1(x)=\frac{8}{35}+\frac{32}{35}x.$$

Misol 8. $y=|x|$ funksiyani $[-1;1]$ oraliqda Chebishev birinchi tur ko'phadlari orqali o'rtacha kvadratik ma'noda yaqinlashtiring.

Yechish. Berilgan oraliqda $y=|x|$ funksiya juft bo'lganligi uchun quyidagicha

$$|x|=\sum_{n=0}^{\infty}a_{2n}T_{2n}(x)=\sum_{n=0}^{\infty}a_{2n}\cos(2n\arccos x)$$

ko'rinish o'rinlidir. Noma'lum koeffitsiyentlar quyidagicha aniqlanadi:

$$a_{2n}=\frac{(f,T_{2n}(x))}{(T_{2n}(x),T_{2n}(x))},\quad n=0,1,\ldots,$$

$$(T_0(x),T_0(x))=\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}=\arcsin x\Big|_{-1}^{1}=\frac{\pi}{2}+\frac{\pi}{2}=\pi,$$

$$(f,T_0(x))=\int_{-1}^{1}\frac{|x|}{\sqrt{1-x^2}}dx=2\int_{0}^{1}\frac{x}{\sqrt{1-x^2}}dx=-2\sqrt{1-x^2}\Big|_0^1=2,$$

$$a_0=\frac{2}{\pi},$$

$$(T_{2n}(x),T_{2n}(x))=\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\cos^2(2n\arccos x)dx=\begin{Bmatrix}\arccos x=t\\ dx=-\sin t dt\\ x=1,\ t=0\\ x=-1,\ t=\pi\end{Bmatrix}=$$

$$=\int_0^{\pi}\cos^2 2nt dt=\int_0^{\pi}\left(\frac{1+\cos 4nt}{2}\right)dt=\frac{\pi}{2},$$

$$(f,T_{2n}(x))=\int_{-1}^{1}\frac{|x|}{\sqrt{1-x^2}}\cos(2n\arccos x)dx=2\int_0^1\frac{x}{\sqrt{1-x^2}}\cos(2n\arccos x)dx=$$

$$=\begin{Bmatrix}x=\cos t\\ dx=-\sin t dt\\ x=0,\ t=\dfrac{\pi}{2}\\ x=1,\ t=0\end{Bmatrix}=2\int_0^{\frac{\pi}{2}}\frac{\cos t\cdot\cos 2nt}{\sin t}\sin t dt=2\int_0^{\frac{\pi}{2}}\cos t\cdot\cos 2nt dt=$$

$$=\int_0^{\frac{\pi}{2}}\left[\cos(2n+1)t+\cos(2n-1)t\right]dt=\left(\frac{1}{2n+1}\sin(2n+1)t+\frac{1}{2n-1}\sin(2n-1)t\right)\Bigg|_0^{\frac{\pi}{2}}=$$

$$=\frac{1}{2n+1}\sin(2n+1)\cdot\frac{\pi}{2}+\frac{1}{2n-1}\sin(2n-1)\cdot\frac{\pi}{2}=\frac{(-1)^n}{2n+1}+\frac{(-1)^{n+1}}{2n-1}=$$

$$=\frac{(-1)^{n+1}(-2n+1+2n+1)}{4n^2-1}=\frac{(-1)^{n+1}\cdot 2}{4n^2-1}.$$

Demak,

$$a_{2n}=\frac{(-1)^{n+1}\cdot 2}{4n^2-1}:\frac{\pi}{2}=\frac{4(-1)^{n+1}}{\pi(4n^2-1)},\quad n=1,2...$$

bo'ladi. Natijada quyidagiga ega bo'lamiz:

$$|x|=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cdot T_{2n}(x).$$

Bu tenglikda $x=\pm 1$ desak va $T_{2n}(\pm 1)=1$ ekanligini e'tiborga olsak,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}=\frac{\pi}{4}-\frac{1}{2}$$

hosil bo'ladi.

Misol 9. Oldingi misolda eng kam og'ish miqdorini aniqlang.

Yechish. Eng kam og'ish

$$\delta_n^2=\int_{-1}^{1}\frac{f^2(x)}{\sqrt{1-x^2}}dx-\frac{\pi}{2}\left[2a_0^2+a_1^2+...+a_n^2\right]$$

formula bilan aniqlanadi.

$$\int_{-1}^{1}\frac{(|x|)^2}{\sqrt{1-x^2}}dx=\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}dx=\frac{\pi}{2},$$

$$\delta_n^2=\frac{\pi}{2}-\frac{\pi}{2}\left[2\cdot\left(\frac{2}{\pi}\right)^2+\left(\frac{4(-1)^{n+1}}{\pi(4n^2-1)}\right)^2\right]=$$

$$=\frac{\pi}{2}-\frac{\pi}{2}\left[\frac{8}{\pi^2}+\frac{16}{\pi^2(4n^2-1)^2}\right]=\frac{\pi}{2}-\frac{4}{\pi}\left[1+\frac{2}{(4n^2-1)^2}\right].$$

Misol 10. Funksiya jadval bilan berilgan:

x	-2	-1	1	2
$f(x)$	4	2	2	4

Uni ikkinchi tartibli algebraik ko'phad bilan yaqinlashtiring.

Yechish. Quyidagi jadvalni tuzamiz:

x^0	x	x^2	x^3	x^4	f	$f{\cdot}x$	$f{\cdot}x^2$
1	–2	4	–8	16	4	–8	16
1	–1	1	–1	1	2	–2	2
1	1	1	1	1	2	2	2
1	2	4	8	16	4	8	16
S_0	S_1	t_2	S_3	S_4	t_0	t_1	t_2
4	0	10	0	34	12	0	36

Bundan quyidagi

$$\begin{cases} S_0a_0 + S_1a_1 + S_2a_2 = t_0 \\ S_1a_0 + S_2a_1 + S_3a_2 = t_1 \\ S_2a_0 + S_3a_1 + S_4a_2 = t_2 \end{cases}$$

tenglamalar sistemasini tuzamiz va jadvaldagi qiymatlarini qo'yamiz.

$$\left.\begin{aligned} 4a_0 + 0\cdot a_1 + 10a_2 = 12 \\ 0\cdot a_0 + 10a_1 + 0\cdot a_2 = 0 \\ 10a_0 + 0\cdot a_1 + 34a_2 = 36 \end{aligned}\right\} \Rightarrow a_1 = 0,$$

$$\left.\begin{aligned} 2a_0 + 5a_2 = 6 \\ 5a_0 + 17a_2 = 18 \end{aligned}\right\} \Rightarrow 9a_2 = 6, \quad a_2 = \frac{2}{3},$$

$$2a_0 = 6 - 5\cdot\frac{2}{3} = \frac{18-10}{3} = \frac{8}{3}, \quad a_0 = \frac{4}{3},$$

$$P_2(x) = \frac{4}{3} + \frac{2}{3}\cdot x^2.$$

Misol 11. Quyida $f(x)$ funksiya jadval ko'rinishida berilgan. Interpolyatsion kubik splayn quring va funksiyaning $x=0,22;\ 0,25;\ 0,28;\ 0,31;\ 0,35$ dagi qiymatlarini hisoblang.

i	0	1	2	3	4	5
x_i	0,2	0,24	0,27	0,30	0,32	0,38
$f(x_i)$	1,2214	1,2710	1,3100	1,3499	1,3771	1,4623

$2M_0-0,1M_1=2,5699,\quad 0,3M_4+2M_5=3,3378$. Bu yerda $M_i=S_3''(x_i)$

Yechish. $h_1=0,04;\ h_2=0,03;\ h_3=0,02;\ h_4=0,02;\ h_5=0,06$ ekanligi jadvaldan ko'rinib turibdi. $M_i\left(i=\overline{0,5}\right)$ larni qiymatlarini haydash usuli bilan topamiz: $M_0=1,387$; $M_1=2,032$; $M_2=0,670$; $M_3=1,289$; $M_4=1,550$; $M_5=1,436$. (4), (2.8-§) bo'yicha interpolyatsion kubik splaynlar qurish uchun barcha ma'lumotlarga ega bo'ldik. Endi har bir oraliq uchun kubik splaynlarni quramiz.

1. $i=1$, ya'ni $[0,2;0,24]$ oraliq uchun:

$$S_3(x)=M_0\frac{(x_1-x)^3}{6h_1}+M_1\frac{(x-x_0)^3}{6h_1}+\left(f_0-\frac{M_0h_1^2}{6}\right)\frac{x_1-x}{h_1}+\left(f_1-\frac{M_1h_1^2}{6}\right)\frac{x-x_0}{h_1}\approx$$

$$\approx\frac{1,387}{0,24}(0,24-x)^3+\frac{2,032}{0,24}(x-0,2)^3+\left(1,2214-\frac{1,387\cdot 0,04^2}{6}\right)\frac{0,24-x}{0,04}+$$

$$+\left(1,2712-\frac{2,032\cdot 0,04^2}{6}\right)\frac{x-0,2}{0,04},$$

$$S_3(x)\approx 2,688x^3-0,919x^2+1,257x+0,985.$$

$$S_3(0,22)\approx 2,688\cdot 0,22^3-0,919\cdot 0,22^2+1,257\cdot 0,22+0,985.$$

$$S_3(0,22)\approx 0,029-0,044+0,277+0,985\approx 1,24654.$$

2. $i=2$, ya'ni $[0,24;0,27]$ oraliq uchun:

$$S_3(x)=M_1\frac{(x_2-x)^3}{6h_2}+M_2\frac{(x-x_1)^3}{6h_2}+\left(f_1-\frac{M_1h_2^2}{6}\right)\frac{x_2-x}{h_2}+\left(f_2-\frac{M_2h_2^2}{6}\right)\frac{x-x_1}{h_2}\approx$$

$$\approx 11,289\cdot(0,27-x)^3+3,722\cdot(x-0,24)^3+$$
$$+42,363\cdot(0,27-x)+43,663\cdot(x-0,24),$$
$$S_3(x)\approx -7,567x^3+6,464x^2-0,526x+1,13.$$

$$S_3(0,25)\approx -7,567\cdot 0,25^3+6,464\cdot 0,25^2-0,526\cdot 0,25+1,13.$$
$$S_3(0,25)\approx -0,118+0,404-0,132+1,13\approx 1,284.$$

3. $i=3$, ya'ni $[0,27;0,30]$ oraliq uchun:

$$S_3(x)=M_2\frac{(x_3-x)^3}{6h_3}+M_3\frac{(x-x_2)^3}{6h_3}+\left(f_2-\frac{M_2h_3^2}{6}\right)\frac{x_3-x}{h_3}+\left(f_3-\frac{M_3h_3^2}{6}\right)\frac{x-x_2}{h_3}\approx$$

$$\approx\frac{0,67}{6\cdot 0,03}(0,3-x)^3+\frac{1,289}{6\cdot 0,03}(x-0,27)^3+\left(1,31-\frac{0,67\cdot 0,03^2}{6}\right)\frac{0,3-x}{0,03}+$$
$$+\left(1,3499-\frac{1,289\cdot 0,03^2}{6}\right)\frac{x-0,27}{0,03},$$

$$S_3(x)\approx 3,439x^3-2,45x^2+1,888x+0,911.$$

$$S_3(0,28)\approx 3,439\cdot 0,28^3-2,45\cdot 0,28^2+1,888\cdot 0,28+0,911.$$
$$S_3(0,28)\approx 0,075-0,192+0,529+0,911\approx 1,3226.$$

4. $i=4$, ya'ni $[0,3;0,32]$ oraliq uchun:

$$S_3(x)=M_3\frac{(x_4-x)^3}{6h_4}+M_4\frac{(x-x_3)^3}{6h_4}+\left(f_3-\frac{M_3h_4^2}{6}\right)\frac{x_4-x}{h_4}+\left(f_4-\frac{M_4h_4^2}{6}\right)\frac{x-x_3}{h_4}\approx$$

$$\approx\frac{1,289}{6\cdot 0,02}(0,32-x)^3+\frac{1,55}{6\cdot 0,02}(x-0,3)^3+\left(1,3499-\frac{1,289\cdot 0,02^2}{6}\right)\frac{0,32-x}{0,02}+$$
$$+\left(1,3771-\frac{1,55\cdot 0,02^2}{6}\right)\frac{x-0,3}{0,02},$$

$$S_3(x)\approx 2,175\cdot x^3-1,313\cdot x^2+1,547\cdot x+0,945,$$

$$S_3(0,31)\approx 2,175\cdot 0,31^3-1,313\cdot 0,31^2+1,547\cdot 0,31+0.945,$$

$$S_3(0,31)\approx 0,065-0,126+0,48+0,945\approx 1,3636.$$

5. $i=5$, ya'ni $[0,32;0,38]$ oraliq uchun:

$$S_3(x)=M_4\frac{(x_5-x)^3}{6h_5}+M_5\frac{(x-x_4)^3}{6h_5}+\left(f_4-\frac{M_4h_5^2}{6}\right)\frac{x_5-x}{h_5}+\left(f_5-\frac{M_5h_5^2}{6}\right)\frac{x-x_4}{h_5}\approx$$

$$\approx\frac{1,55}{6\cdot 0,06}(0,38-x)^3+\frac{1,436}{6\cdot 0,06}(x-0,32)^3+\left(1,3771-\frac{1,55\cdot 0,06^2}{6}\right)\frac{0,38-x}{0,06}+$$

$$+\left(1,4623-\frac{1,436\cdot 0,06^2}{6}\right)\frac{x-0,32}{0,06},$$

$$S_3(x)\approx -0,317\cdot x^3+1,08\cdot x^2+0,781\cdot x+1,027,$$

$$S_3(0,35)\approx -0,317\cdot 0,35^3+1,08\cdot 0,35^2+0,781\cdot 0,35+1,027,$$

$$S_3(0,35)\approx -0,014+0,132+0,273+1,027\approx 1,4184.$$

Misollar.

1. Funksiya jadval bilan berilgan:

1) jadval qadami teng emas;

2) jadval qadami teng.

Lagranj interpolyatsion ko'phadini ishlatib argumentning berilgan qiymatida funksiya qiymatini taqriban toping.

1-vazifaga doir jadvallar

1-jadval

x	y
0,43	1,63597
0,48	1,73234
0,55	1,87686
0,62	2,03345
0,70	2,22846
0,75	2,35973

Variant №	x
1	0,702
6	0,512
11	0,645
16	0,736
21	0,608

5-jadval

x	y
0,68	0,80866
0,73	0,89492
0,80	1,02964
0,88	1,20966
0,93	1,34087
0,99	1,52368

Variant №	x
5	0,896
10	0,812
15	0,774
20	0,955
25	0,715

2-vazifaga doir jadvallar

1-jadval

x	y
1,375	5,04192
1,380	5,17744
1,385	5,32016
1,390	5,47069
1,395	5,62968
1,400	5,79788

Variant №	x
1	1,3832
6	1,3926
11	1,3862
16	1,3934
21	1,3866

2-jadval

x	y
0,115	8,65729
0,120	8,29329
0,125	7,95829
0,130	7,64893
0,135	7,36235
0,140	7,09613

Variant №	x
2	0,1264
7	0,1315
12	0,1232
17	0,1334
22	0,1285

2-jadval

x	y
0,02	1,02316
0,08	1,09590
0,12	1,14725
0,17	1,21483
0,23	1,30120
0,30	1,40976

Variant №	x
2	0,102
7	0,114
12	0,125
17	0,203
22	0,154

3-jadval

x	y
0,35	2,73951
0,41	2,30080
0,47	1,96864
0,51	1,78776
0,56	1,59502
0,64	1,34310

Variant №	x
3	0,526
8	0,453
13	0,482
18	0,552
23	0,436

4-jadval

x	y
0,41	2,57418
0,46	2,32513
0,52	2,09336
0,60	1,86203
0,65	1,74926
0,72	1,62098

Variant №	x
4	0,616
9	0,478
14	0,665
19	0,537
24	0,673

3-jadval

x	y
0,150	6,61659
0,155	6,39989
0,160	6,19658
0,165	6,00551
0,170	5,82558
0,175	5,65583

Variant №	x
3	0,1521
8	0,1611
13	0,1662
18	0,1542
23	0,1625

4-jadval

x	y
0,180	5,61543
0,185	5,46693
0,190	5,32634
0,195	5,19304
0,200	5,06649
0,205	4,94619

Variant №	x
4	0,1838
9	0,1875
14	0,1944
19	0,1976
24	0,2038

5-jadval

x	y
0,210	4,83170
0,215	4,72261
0,220	4,61855
0,225	4,51919
0,230	4,42422
0,235	4,33337

Variant №	x
5	0,2121
10	0,2165
15	0,2232
20	0,2263
25	0,2244

2. 1-vazifa. Chiziqli interpolyatsiyani qo'llab, funksiyaning berilgan nuqtadagi qiymatini toping. Buning uchun Bradis jadvalidan oltita qiymat oling, chekli ayirmalar jadvalini tuzing va chiziqli interpolyatsiyalashni qo'llash mumkinligiga ishonch hosil qiling.

№1. a) $\sin 0,1436$; b) $\cos 1,1754$.

№2. a) $\sin 0,2453$; b) $\cos 1,0938$.

№3. a) $\sin 0,4456$; b) $\cos 1,0045$.

№4. a) $\sin 0,6235$; b) $\cos 0,9464$.

№5. a) $\sin 0,7243$; b) $\cos 0,8675$.

№6. a) $\sin 0,8453$; b) $\cos 0,4324$.

№7. a) $\sin 0,9675$; b) $\cos 0,3436$.

№8. a) $\mathrm{tg}\, 0,4052$; b) $\cos 0,7645$.

№9. a) $\mathrm{tg}\, 0,4527$; b) $\cos 0,7466$.

№11. a) $\mathrm{tg}\, 0,3083$; b) $\cos 0,8235$.

№12. a) $\mathrm{tg}\, 0,3864$; b) $\cos 0,9222$.

№13. a) $\mathrm{tg}\, 0,3224$; b) $\cos 0,8465$.

№14. a) $\sin 1,0236$; b) $\cos 0,2267$.

№15. a) $\sin 0,9057$; b) $\cos 0,2632$.

2-vazifa. Funksiya jadval bilan berilgan. Kvadratik interpolyatsiyani qo'llab, funksiyaning berilgan nuqtadagi qiymatini aniqlang. Kvadratik interpolyatsiyalashni qo'llash mumkinligini aniqlang.

x	y
1,675	9,5618
1,676	9,4703
1,677	9,3804

Variant №	x
1	1,6763
2	1,6778
3	1,6785

1,678	9,2923	4	1,6794
1,679	9,2057	5	1,6801
1,680	9,1208	6	1,6816
1,681	9,0373	7	1,6822
1,682	8,9554	8	1,6837
1,683	8,8749	9	1,6849
1,684	8,7959	10	1,6853
1,685	8,7182	11	1,6868
1,686	8,6418	12	1,6773
1,687	8,5668	13	1,6788
1,688	8,4931	14	1,6813
		15	1,6845

3. $f(x)$ funksiyaning qiymatlari quyidagi jadvallarda berilgan. Kubik interpolyatsion splayn quring va ko'rsatilgan nuqtalarda funksiyaning qiymatini hisoblang. Hisoblashlarda $M_i = S_3''(x_i)$ deb hisoblang.

1.

i	0	1	2	3	4	5
x_i	0,1	0,15	0,19	0,25	0,28	0,30
$f(x_i)$	1,1052	1,1618	1,2092	1,2840	1,3231	0,3499

$$2M_0 + M_1 = 3,3722, \quad 0,5M_4 + 2M_5 = 3,3614, \quad x = 0,20.$$

2.

i	0	1	2	3	4	5
x_i	0,2	0,24	0,26	029	0,32	0,38
$f(x_i)$	1,2214	1,2712	1,2969	1,3364	1,3771	1,4623

$$2M_0 + 0,1M_1 = 2,5699, \quad 0,3M_4 + 2M_5 = 3,3378, \quad x = 0,31.$$

3.

i	0	1	2	3	4	5
x_i	0,1	0,13	0,17	0,20	0,25	0,28
$f(x_i)$	0,0998	0,1296	0,1692	0,1987	0,2474	0,2764

$$2M_0 + 0{,}5M_1 = -0{,}2644, \quad 0{,}4M_4 + 2M_5 = -0{,}6580, \quad x = 0{,}15.$$

4.

i	0	1	2	3	4	5
x_i	0,1	0,15	0,18	0,22	0,28	0,30
$f(x_i)$	1,1052	1,1618	1,1972	1,2461	1,3231	0,3499

$$2M_0 + M_1 = 3{,}3722, \quad 0{,}5M_4 + 2M_5 = 3{,}3614, \quad x = 0{,}16.$$

5.

i	0	1	2	3	4	5
x_i	0,2	0,24	0,27	0,30	0,32	0,38
$f(x_i)$	1,2214	1,2710	1,3100	1,3499	1,3771	1,4623

$$2M_0 + 0{,}1M_1 = 2{,}5699, \quad 0{,}3M_4 + 2M_5 = 3{,}3378. \quad x = 0{,}26.$$

III BOB. INTEGRALLARNI TAQRIBIY HISOBLASH

Aniq integralni hisoblashda qo'llaniladigan

$$\int_a^b p(x)f(x)dx \cong \sum_{k=0}^{n} A_k f(x_k) \qquad (1)$$

taqribiy tenglik *kvadratur formula* deb ataladi. Bu yerda $p(x) \geq 0$ bo'lib, uni odatda vazn funksiya, x_k va A_k $(k = 0,1,...,n)$ lar mos ravishda *kvadratur formulaning tugun nuqtalari* hamda *koeffitsiyentlari* deyiladi.

$$R_n(f) = \int_a^b p(x)f(x)dx - \sum_{k=0}^{n} A_k f(x_k) \qquad (2)$$

kvadratur formulaning qoldiq hadi deyiladi.

Ta'rif. Agar (1) kvadratur formula m-darajali ixtiyoriy algebraik ko'phadlar uchun aniq bo'lib, $f(x) = x^{m+1}$ uchun aniq bo'lmasa, u holda uning *algebraik aniqlik darajasi m* ga teng deyiladi.

Biz quyida algebraik aniqlik darajasiga ega bo'lgan kvadratur formulalar bilangina tanishamiz.

3.1-§. Interpolyatsion kvadratur formulalar

Faraz qilaylik, $[a,b]$ oraliqda o'zi va $n+1$ tartibgacha hosilalari uzluksiz bo'lgan $f(x)$ funksiyadan $p(x) \geq 0$ vazn funksiya bilan olingan integralni taqribiy hisoblash lozim bo'lsin. Buning uchun $[a,b]$ ga tegishli va turli bo'lgan x_k, $k = 0,1,...,n$ tugun nuqtalar olib $f(x)$ funksiyaning n-tartibli Lagranj interpolyatsion ko'phadini tuzamiz, ya'ni

$$f(x) = L_n(x) + r_n(f, x) \qquad (3)$$

bu yerda $r_n(f,x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$ – Lagranj interpolyatsion ko'phadining qoldiq hadi. (3) tenglikning ikki tomonini $p(x)$ vazn funksiyaga ko'paytirib, $[a,b]$ oraliq bo'yicha integrallasak,

$$\int_a^b p(x)f(x)dx = \int_a^b p(x)\sum_{k=0}^{n} \frac{\omega_{n+1}(x)}{(x-x_k)\omega'_{n+1}(x_k)} f(x_k)dx + \int_a^b p(x)r_n(f,x)dx$$

ni hosil qilamiz. Agar interpolyatsiyalash yetarlicha yaxshi o'tkazilgan bo'lsa, $r_n(f,x)$ $x \in [a,b]$ uchun kichik miqdordir, undan olingan integralning qiymatini ham kichkina deb, tashlab yuborsak

$$\int_a^b p(x)f(x)dx \cong \sum_{k=0}^{n} A_k f(x_k) \qquad (4)$$

kvadratur formulaga ega bo'lamiz. Bunda

$$A_k = \int_a^b p(x)\frac{\omega_{n+1}(x)}{(x-x_k)\omega'_{n+1}(x_k)}dx, \quad k = 0,1,\dots,n$$

Yuqorida ko'rsatilgan tartibda hosil qilingan (4) formula, odatda, *interpolyatsion kvadratur formula* deyiladi va uning algebraik aniqlik darajasi n ga teng. Uning qoldiq hadi

$$R_n(f) = \frac{1}{(n+1)!}\int_a^b p(x)\omega_{n+1}(x)\cdot f^{(n+1)}(\xi)dx$$

ko'rinishga ega. Bunda $\omega_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$.

Eng sodda kvadratur formulalar bilan tanishamiz. Bu yerda $p(x) \equiv 1$.

1. $n = 0$, $x_0 = \frac{a+b}{2}$, $\omega_1(x) = x - x_0$

$$A_0 = \int_a^b dx = b - a,$$

$\int_a^b f(x)dx \cong (b-a)f\left(\frac{a+b}{2}\right)$ *o'rta to'g'ri to'rtburchaklar formulasi.*

Uning qoldiq hadi

$$R_0(f)=\int_a^b f(x)dx-(b-a)f\left(\frac{a+b}{2}\right)$$

ni topish uchun $f(x)$ ni $[a,b]$ da ikkinchi tartibli uzluksiz hosilaga ega deb faraz qilamiz. U holda Teylor formulasiga ko'ra:

$$f(x)-f\left(\frac{a+b}{2}\right)=\left(x-\frac{a+b}{2}\right)f'\left(\frac{a+b}{2}\right)+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2 f''(\xi),$$

bu yerda $x\le\xi(x)\le\frac{a+b}{2}$. Bu tenglikning har ikkala tomonini a dan b gacha integrallasak,

$$R_0(f)=\frac{1}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^2 f''(\xi)dx \qquad (5)$$

kelib chiqadi, chunki $\int_a^b\left(x-\frac{a+b}{2}\right)f'\left(\frac{a+b}{2}\right)dx=0$.

Integral ostidagi $\left(x-\frac{a+b}{2}\right)^2$ funksiya o'z ishorasini saqlaydi, shuning uchun (5) integralga umumlashgan o'rta qiymat haqidagi teoremani qo'llash mumkin:

$$R_0(f)=\frac{L}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^2 dx=L\cdot\frac{(b-a)^3}{24}, \qquad (6)$$

bunda $m\le L\le M$, $m=\min\limits_{[a,b]} f''(x)$, $M=\max\limits_{[a,b]} f''(x)$, $f''(x)$ uzluksiz bo'lganligi uchun Koshi teoremasiga ko'ra shunday $\xi\in[a,b]$ mavjudki,

$$L=f''(\xi).$$

Endi (6)

$$R_0(f)=\frac{(b-a)^3}{24}f''(\xi) \qquad (7)$$

ko'rinishga ega bo'ladi.

Qoldiq had bahosi: $|R_0(f)|\le \frac{(b-a)^3}{24}\max\limits_{[a,b]}|f''(x)|$.

2. $n=1,\ x_0=a,\ \ x_1=b,\ \ \omega_2(x)=(x-a)(x-b),$

$$A_0=\int_a^b\frac{x-b}{a-b}dx=\frac{b-a}{2},\quad A_1=\int_a^b\frac{x-a}{a-b}dx=\frac{b-a}{2}.$$

$\int_a^b f(x)dx\cong\frac{(b-a)}{2}\big(f(a)+f(b)\big)$ – *trapetsiya formulasi*,

$R_1(f)=\frac{1}{2}\int_a^b(x-a)(x-b)f''(\xi)dx$.

$[a,b]$ oraliqda $(x-a)(x-b)\le 0$ bo'lganligi uchun o'rta qiymat haqidagi umumlashgan teoremani qo'llasak,

$$R_1(f)=\frac{1}{2}f''(\xi)\int_a^b(x-a)(x-b)dx=-\frac{(b-a)^3}{12}f''(\xi) \qquad (8)$$

bo'ladi, bunda $\xi\in[a,b]$.

Qoldiq had bahosi: $|R_1(f)|\le\frac{(b-a)^3}{12}\max\limits_{[a,b]}|f''(x)|$.

3. $n=2,\ x_0=a,\ \ x_1=\frac{a+b}{2},\ x_2=b,\ \omega_3(x)=(x-a)\left(x-\frac{a+b}{2}\right)(x-b),$

$$A_0=\int_a^b\frac{\left(x-\frac{a+b}{2}\right)(x-b)}{\left(\frac{a-b}{2}\right)(a-b)}dx=\frac{b-a}{6},\qquad A_1=\int_a^b\frac{(x-a)(x-b)}{\frac{b-a}{2}\frac{a-b}{2}}dx=\frac{2}{3}(b-a)$$

$$A_2=\int_a^b\frac{\left(x-\frac{a+b}{2}\right)(x-a)}{\left(\frac{b-a}{2}\right)(b-a)}dx=\frac{b-a}{6}.$$

$$\int_a^b f(x)\,dx \cong \frac{(b-a)}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b) \right)$$ – *Simpson kvadratur formulasi*. Uning qoldiq hadini keltiramiz:

$$R_2(f) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi), \quad a \le \xi \le b.$$

Qoldiq had bahosi:

$$|R_2(f)| \le \frac{(b-a)^5}{2880} \max_{[a,b]} \left| f^{(4)}(x) \right|.$$

4. *Umumlashgan kvadratur formulalar.* Taqribiy integrallash formulasining xatoligini kamaytirish maqsadida amaliyotda umumlashgan kvadratur formulalardan foydalaniladi. Buning uchun $[a,b]$ oraliqni $h = \frac{b-a}{N}$ uzunlikda teng N bo'lakka bo'lamiz. Har bir $[x_k, x_{k+1}]$ qismiy oraliq uchun o'rta to'g'ri to'rtburchaklar formulasini qo'llab, ularni $k = 0,1,\dots,N-1$ lar bo'yicha yig'ib chiqsak, umumlashgan o'rta to'g'ri to'rtburchaklar formulasi kelib chiqadi:

$$\int_a^b f(x)\,dx = \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} f(x)\,dx \cong$$
$$\cong \frac{b-a}{N}\left(f\left(a+\frac{h}{2}\right) + f\left(a+\frac{3h}{2}\right) + \dots + f\left(a+\frac{(2N-1)h}{2}\right) \right) \quad (9)$$

Buning qoldiq hadi $R_{0.um}(f)$ ni esa har bir qismiy oraliq uchun o'rta to'g'ri to'rtburchak formulasining qoldiq hadlari yig'indisiga teng bo'ladi:

$$R_{0.um}(f) = \frac{(b-a)^3}{24N^3} \sum_{k=0}^{N-1} f''(\xi_k) \quad (10)$$

bu yerda $x_k \le \xi_k \le x_{k+1}$.

Ikkinchi tartibli hosilaning uzluksizligidan, Koshi teoremasiga binoan shunday $\xi \in [a,b]$ mavjudki,

$$\frac{1}{N}\sum_{k=0}^{N-1} f''(\xi_k) = f''(\xi)$$

bo'ladi. Demak, (10) quyidagicha bo'ladi

$$R_{0.y.u}(f) = \frac{(b-a)^3}{24N^2} f''(\xi) = \frac{1}{N^2} R_0(f)$$

Qoldiq hadi bahosi:

$$\left|R_{0.um}(f)\right| \le \frac{1}{N^2}\left|R_0(f)\right|.$$

Agar har bir $[x_k, x_{k+1}]$ qismiy oraliq uchun trapetsiya formulasini qo'llasak, umumlashgan trapetsiyalar formulasiga ega bo'lamiz:

$$\int_a^b f(x)\,dx \cong \frac{(b-a)}{N}\left[f(x_0) + f(x_N) + 2\sum_{i=1}^{N-1} f(x_i) \right].$$

Uning qoldiq hadi

$$R_{1.um}(f) = -\frac{(b-a)^3}{12N^2} f''(\xi), \quad a \le \xi \le b,$$

qoldiq hadning bahosi esa

$$\left|R_{1.um}(f)\right| \le \frac{1}{N^2}\left|R_1(f)\right|$$

ko'rinishga ega.

Endi $[a,b]$ oraliqni teng $2N$ bo'lakka bo'lamiz va har bir $[x_{2k-2}, x_{2k}]$ $k = 1, 2, \ldots, N$ qismiy oraliqqa Simpson formulasini qo'llasak,

$$\int_a^b f(x)\,dx = \sum_{k=1}^{N} \int_{x_{2k-2}}^{x_{2k}} f(x)\,dx \cong$$

$$\cong \frac{(b-a)}{6N}\left[f(a) + f(b) + 4\sum_{k=1}^{N} f(x_{2k-1}) + 2\sum_{k=1}^{N-1} f(x_{2k}) \right]$$

hosil bo'ladi. Bu formulaning qoldiq hadi

$$R_{2.um}(f) = -\frac{(b-a)^5}{2880N^4} f^{(4)}(\xi), \quad a \le \xi \le b,$$

uning bahosi

$$\left|R_{2,umr}(f)\right| \le \frac{1}{N^4}\left|R_2(f)\right|$$

ko'rinishga ega.

3.2-§. Gauss tipidagi kvadratur formula

Quyidagi kvadratur formulani qaraymiz:

$$\int_a^b p(x) f(x) dx \cong \sum_{k=1}^n A_k f(x_k), \qquad (1)$$

bu yerda $p(x) \ge 0$ vazn funksiya, A_k, x_k, $k = 0,1,...,n$ noma'lumdir. Bu noma'lumlarni shunday aniqlash lozimki, (1) ning algebraik aniqlik darajasi $2n-1$ ga teng bo'lsin. Quyidagi teorema o'rinlidir.

Teorema. (1) kvadratur formulaning algebraik aniqlik darajasi $2n-1$ ga teng bo'lishligi uchun uning tugun nuqtalari $[a,b]$ da $p(x) \ge 0$ vazn funksiya bilan n-darajali ortogonal ko'phadning ildizlari bo'lishligi zarur va yetarlidir.

Isboti. Zaruriyligi. Faraz qilaylik, (1) ning algebraik aniqlik darajasi $2n-1$ bo'lsin. Tugun nuqtalarni turli deb hisoblasak, (1) ning interpolyatsiyaligi ta'minlanadi. Teoremadagi ortogonal ko'phadni $P_n(x)$ deb belgilaylik. Darajasi n dan kichik bo'lgan ixtiyoriy ko'phad $Q(x)$ ni olib, $f(x) = P_n(x)Q(x)$ deylik. Bu ko'phadning darajasi $2n-1$ dan ortmaydi. Shuning uchun ham uni (1) formula aniq integrallaydi:

$$\int_a^b p(x) P_n(x) Q(x) dx \cong \sum_{k=1}^n A_k P_n(x_k) Q(x_k).$$

Bu yerda, shartga ko'ra, integral nolga teng, o'ng tomon ham nolga teng bo'lishligi uchun $P_n(x_k) = 0$, $k = 1,2,...,n$ shartlar bajarilishi kerak. $Q(x_k)$ k ning barcha qiymatlari uchun nolga aylanmaydi, chunki u darajasi n dan kichik ixtiyoriy ko'phad. Demak, $P_n(x_k)=0,\ k=1,2,...,n$ bo'lsa, yuqoridagi tenglik bajariladi.

Yetarliligi. Faraz qilaylik, (1) interpolyatsion va $P_n(x)$ $p(x)\geq 0$ vazn bilan ortogonal ko'phad bo'lsin.

Endi (1) ning algebraik aniqlik darajasi $2n-1$ ligini ko'rsatamiz. Agar $f(x)$ darajasi $2n-1$ dan katta bo'lmagan ko'phad bo'lsa, uni

$$f(x)=P_n(x)Q(x)+R(x) \tag{2}$$

ko'rinishda yozish mumkin, bu yerda $Q(x)$ va $R(x)$ larning darajasi n dan kichik. Bu tenglikning ikkala tomonini $p(x)$ ga ko'paytirib, a dan b gacha integrallaymiz:

$$\int_a^b p(x)f(x)dx=\int_a^b p(x)P_n(x)Q(x)dx+\int_a^b p(x)R(x)dx.$$

Shartga ko'ra, o'ng tomondagi birinchi integral nolga teng, ikkinchi integraldagi $R(x)$ darajasi n dan kichik ko'phad bo'lganligi, (1) ning interpolyatsionligi va (2) dan $f(x_k)=R(x_k)$ ekanligini e'tiborga olsak,

$$\int_a^b p(x)f(x)dx=\sum_{k=1}^{n}A_k f(x_k)$$

kelib chiqadi. Shu bilan yetarlilik sharti ham isbotlandi.

Endi ortogonal ko'phadning nollari haqidagi teoremani ko'ramiz.

Teorema. $[a,b]$ oraliqda $p(x)\geq 0$ vazn funksiya bilan ortogonal bo'lgan $P_n(x)$ ko'phadning barcha ildizlari haqiqiy, turli va (a,b) intervalga tegishli.

Isboti. $P_n(x)$ ning (a,b) ga tegishli toq karrali ildizlarini $\xi_1,\xi_2,...,\xi_m$ deb belgilaylik. Teorema isbotlanishi uchun $m=n$ ekanligini ko'rsatish kifoyadir, chunki bundan $P_n(x)$ ko'phadning boshqa ildizlari yo'qligi va ularning turliligi kelib chiqadi. Teskarisini faraz qilaylik, ya'ni $m<n$ bo'lsin. Ushbu

$$Q_m(x) = \sum_{i=1}^{m}(x - \xi_i)$$

ko'phadni tuzamiz. $m < n$ bo'lganligi uchun

$$\int_a^b p(x)\cdot P_n(x)\cdot Q_m(x)dx = 0$$

tenglik o'rinli bo'lishi kerak, ammo $P_n(x)\cdot Q_m(x)$ ko'phad $[a,b]$ da ishora saqlagani va $[a,b]$ da chekli nuqtalarda nolga tengligi sababli yuqoridagi integral noldan farqli. Bu ziddiyat teoremani isbotlaydi.

Shunday qilib, (1) ko'rinishga ega bo'lgan kvadratur formula mavjudligini ko'rdik. Ba'zan (1) Gauss tipidagi kvadratur formula deb ham ataladi, chunki Gauss (1) formulani xususiy holda, ya'ni oraliq $[-1,1]$, vazn funksiya $p(x) = 1$ bo'lganda keltirib chiqargan.

Gauss tipidagi kvadratur formula quyidagi xossalarga ega.

1-xossa. (1) kvadratur formulaning tugun nuqtalari va koeffitsiyentlari har qanday tanlanganda ham (1) ning algebraik aniqlik darajasi ortmaydi.

2-xossa. (1) kvadratur formulaning barcha koeffitsiyentlari musbatdir.

3-xossa. Agar $[a,b]$ chekli va $f(x)$ bu oraliqda uzluksiz bo'lsa, u holda Gauss tipidagi kvadratur formula yaqinlashadi:

$$\lim_{n\to\infty} \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}) = \int_a^b p(x)f(x)dx$$

3.3-§. Chebishev tipidagi kvadratur formula

Kvadratur formula

$$\int_a^b p(x)f(x)dx = A\sum_{k=1}^{n} f(x_k) \qquad (1)$$

ko'rinishga ega bo'lsin. Bu yerda $p(x) \geq 0$ vazn funksiya. (1) formulaning noma'lum parametrlari A va $x_k, (k = 1,2,...,n)$ lar bo'lib, ularni shunday topaylikki, (1) ning algebraik aniqlik darajasi n ga teng bo'lsin. Quyidagicha belgilash kiritamiz:

$$\mu_m = \int_a^b p(x)x^m\,dx, \quad m = 0,1,2,... \tag{2}$$

Agar $f(x) = 1$ desak, (1) dan (2) ga asosan

$$A = \frac{\mu_0}{n}$$

bo'ladi. Endi $f(x) = x^m$, $m = 1,2,...,n$ deb, quyidagi nochiziqli tenglamalar sistemasini hosil qilamiz:

$$\sum_{k=1}^{n} x_k^m = \mu_m \cdot \frac{n}{\mu_0}, \quad m = 1,2,...,n\ . \tag{3}$$

(1) kvadratur formulaning tugunlari (3) tenglamalar sistemasining yechimlari bo'lar ekan. (3) tenglamalar sistemasini yechish o'rniga quyidagicha ish tutamiz.

Faraz qilaylik, (1) ning tugun nuqtalari n tartibli

$$\omega_n(x) = \prod_{k=1}^{n}(x - x_k) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n \tag{4}$$

ko'phadning nollari bo'lsin. (4) dan hosila olamiz:

$$\omega_n'(x) = \sum_{i=1}^{n}\frac{\omega_n(x)}{x - x_i} \tag{5}$$

$$\omega_n'(x) = nx^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \cdots + a_{n-1}\ . \tag{6}$$

(5) va (6) ning chap tomonlari bir xil, demak o'ng tomonlari ham teng, ya'ni x ning bir xil darajalari oldidagi koeffitsiyentlar o'zaro teng bo'lishi kerak. Buning uchun (5) ning o'ng tomonidagi bo'lish va yig'ish amalini bajaramiz va $S_m = \sum_{k=1}^{n} x_k^m$, $m = 1,2,...,n$ belgilashni kiritib,

$$
\begin{aligned}
&S_1 + a_1 = 0,\\
&S_2 + a_1 S_1 + 2a_2 = 0,\\
&S_3 + a_1 S_2 + a_2 S_1 + 3a_3 = 0,\\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots\\
&S_n + a_1 S_{n-1} + a_2 S_{n-2} + \dots + n a_n = 0
\end{aligned}
$$

Nyuton munosabatlari deb nomlanadigan formulalarni hosil qilamiz. Bulardan ketma-ket $a_1, a_2, \dots, a_n$ larni aniqlaymiz, so'ng

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0$$

tenglamani yechib, $x_1, x_2, \dots, x_n$ larni topamiz. Agar $x_k, (k = 1, 2, \dots, n)$ lar haqiqiy va turli bo'lsa,

$$\int_a^b p(x) f(x) dx \cong \frac{\mu_0}{n} \sum_{k=1}^{n} f(x_k) \qquad (7)$$

ko'rinishga ega kvadratur formula berilgan n uchun qurilgan bo'ladi va uni *Chebishev tipidagi kvadratur formula* deyiladi. (7) kvadratur formula vazn funksiya $p(x) \equiv 1$, oraliq $[-1,1]$ bo'lganda $n = 1, 2, \dots, 7$ uchun Chebishev parametrlari qiymatini ko'rsatgan.

Eslatib o'tamiz, keyinchalik $n = 8$ va $n \geq 10$ bo'lganda $\omega_n(x) = 0$ tenglama ildizlarining orasida haqiqiy bo'lmaganlarining ham mavjudligi isbotlandi [11].

Bobga tegishli tayanch so'zlar: kvadratur formula, tugun nuqtalar, koeffitsiyentlar, qoldiq had, algebraik aniqlik darajasi, interpolyatsion kvadratur formula, umumlashgan kvadratur formula, Gauss tipidagi kvadratur formula, Chebishev tipidagi kvadratur formula.

Savollar va topshiriqlar

1. Kvadratur formula ta'rifini keltiring.
2. Kvadratur formulaning algebraik aniqlik darajasi.
3. Interpolyatsion kvadratur formula ta'rifi.

4. O'rta to'g'ri to'rtburchaklar formulasi va uning qoldiq hadi.

5. Trapetsiya kvadratur formulasi va uning qoldiq hadi.

6. Simpson kvadratur formulasi va uning qoldiq hadi.

7. Umumlashgan trapetsiya va Simpson kvadratur formulalari va ularning qoldiq had bahosi.

8. Gauss tipidagi kvadratur formula.

9. Ortogonal ko'phadlarning ildizlari haqidagi teorema.

10. Gauss tipidagi kvadratur formula xossalari.

11. Chebishev tipidagi kvadratur formula.

12. Nyuton munosabatlari.

13. Lejandr ortogonal ko'phadlarini aniqlaydigan formulani yozing.

14. Chebishevning birinchi tur va ikkinchi tur ko'phadlarini aniqlaydigan formulani va rekurrent munosabatlarni yozing.

15. $$T_n(x)=\begin{vmatrix} x & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2x & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & \dots & 1 & 2x & 1 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & 2x \end{vmatrix}$$

$$U_n(x)=\begin{vmatrix} 2x & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2x & 1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & \dots & 1 & 2x & 1 \\ 0 & \dots & \dots & \dots & 0 & \dots & 0 & 1 & 2x \end{vmatrix}$$

ekanligini ko'rsating. Determinant tartibi n ga teng.

16. $\int\limits_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx \cong \sum\limits_{i=1}^{n}c_i f(x)$ kvadratur formulaning algebraik aniqlik darajasi eng yuqori bo'lganda $c_1 = c_2 = \ldots = c_n = \frac{\pi}{n}$ bo'lishini ko'rsating.

17. Quyidagi ayniyatlarni isbotlang:

$$2T_n(x)\cdot T_m(x) = T_{n+m}(x) - T_{n-m}(x), \quad n \ge m;$$

$$2(T_n(x))^2 = 1 + T_{2n}(x);$$

$$2(1-x^2)(U_{n-1}(x))^2 = 1 - 2T_{2n}(x);$$

$$2T_n(x)U_{n-1}(x) = U_{2n-1}(x);$$

$$2(x^2-1)U_{m-1}(x)U_{n-1}(x) = T_{n+m}(x) - T_{n-m}(x), \quad n \ge m.$$

Bu yerda $T_k(x)$ va $U_k(x)$ lar mos ravishda Chebishevning birinchi va ikkinchi tur ko'phadlari.

18. $\int\limits_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f(x)dx \cong \frac{4\pi}{2n+1}\sum\limits_{k=1}^{n}\sin^2\frac{k\pi}{2n+1}f\left(\cos\frac{2k\pi}{2n+1}\right)$ kvadratur formulaning algebraik aniqlik darajasi $2n-1$ ekanligini ko'rsating.

Misol 1. $\int\limits_{0}^{1}\frac{dx}{1+x}$ integralni umumlashgan trapetsiya formulasi bilan 10^{-2} aniqlikda hisoblang, qadam h ni qoldiq had bahosidan toping.

Yechish. Umumlashgan trapetsiya formulasini yozamiz:

$$R_{1.um}(f) = -\frac{(b-a)^2}{12N^2}f''(\xi), \qquad a \le \xi \le b$$

Berilgan misolda $a = 0,\ b = 1,\ f(x) = \frac{1}{1+x}$.

$$f'(x) = -\frac{1}{(1+x)^2},\ f''(x) = \frac{2}{(1+x)^3}.$$

$$\left|R_{1,um}(f)\right| \le \frac{1}{12N^2}\max_{[0,1]}\left|f''(x)\right| = \frac{1}{12N^2}\max_{[0,1]}\left|\frac{2}{(1+x)^3}\right| = \frac{1}{12N^2}\cdot 2 = \frac{1}{6N^2},$$

$$\frac{1}{6N^2} \le 10^{-2} \Rightarrow N \ge 5, \ h = 0,2.$$

Demak, $\int_0^1 \frac{dx}{1+x} \cong \frac{1}{10}\left[f(0)+f(1)+2\left(f(0,2)+f(0,4)+f(0,6)+f(0,8)\right)\right] =$

$$= \frac{1}{10}\left[1+\frac{1}{2}+2\left(\frac{1}{1,2}+\frac{1}{1,4}+\frac{1}{1,6}+\frac{1}{1,8}\right)\right] = \frac{1}{10}\cdot 6,9562 \cong 0,69.$$

Misol 2. Yuqoridagi integralni $n=4$ da Chebishev tipidagi kvadratur formula bilan hisoblang.

Yechish. $x = 0,5\cdot(1+t)$ almashtirish bajarsak,

$$I = \int_{-1}^{1}\frac{dt}{3+t}$$

bo'ladi. Bu integralni Chebishev kvadratur formulasi bilan $n=4$ da hisoblaymiz:

$$I = \int_{-1}^{1}\frac{dt}{3+t} \cong \frac{2}{4}\sum_{i=1}^{4}\frac{1}{3+t_i}.$$

Bu yerda t_i, $i=\overline{1,4}$, $\omega_4(t) = t^4 + a_1t^3 + a_2t^2 + a_3t + a_4$ ko'phadning nollari bo'lishi kerak. Uning koeffitsiyentlari Nyuton munosabatlaridan topiladi:

$$\begin{cases} S_1 + a_1 = 0, \\ S_2 + a_1S_1 + 2a_2 = 0, \\ S_3 + a_1S_2 + a_2S_1 + 3a_3 = 0, \\ S_4 + a_1S_3 + a_2S_2 + a_3S_1 + 4a_4 = 0. \end{cases}$$

Bu yerda $S_m = \sum_{k=1}^{4} t_k^m = \mu_m \cdot \frac{4}{\mu_0}$, $m = \overline{1,4}$.

$$\mu_l = \int_{-1}^{1} t^l dt = \left.\frac{x^{l+1}}{l+1}\right|_{-1}^{1} = \frac{1-(-1)^{l+1}}{l+1},$$

$$\mu_0 = 2,\ \mu_1 = 0,\ \mu_2 = \frac{2}{3},\ \mu_3 = 0,\ \mu_4 = \frac{2}{5}.$$

$$S_1 = 0,\ \ S_2 = \frac{4}{3},\ S_3 = 0,\ S_4 = \frac{4}{5}.$$

Bulardan foydalanib, yuqoridagi sistemadan $a_1 = 0$, $a_2 = -\frac{2}{3}$, $a_3 = 0$, $a_4 = \frac{1}{45}$ va $\omega_4(t) = t^4 - \frac{2}{3}t^2 + \frac{1}{45}$ ekanligini aniqlaymiz. Uning ildizlari:

$$t_1 = -\sqrt{\frac{5+2\sqrt{5}}{15}} \cong -0,79465, \quad t_2 = -\sqrt{\frac{5-2\sqrt{5}}{15}} \cong -0,18759,$$

$$t_3 = \sqrt{\frac{5-2\sqrt{5}}{15}} \cong 0,18759, \quad t_4 = \sqrt{\frac{5+2\sqrt{5}}{15}} = 0,79465.$$

Berilgan integralni hisoblaymiz:

$$I = \frac{1}{2}\sum_{i=1}^{4}\frac{1}{3+t_i} = \frac{1}{2}\left(\frac{1}{2,20535} + \frac{1}{2,81241} + \frac{1}{3,18759} + \frac{1}{3,79465}\right) \cong 0,69.$$

Misol 3. $Y = \int_0^1 \frac{dx}{1+x^2}$ integralni Gauss kvadratur formulasi bilan $n = 5$ da hisoblang.

Yechish.

$$x = 0,5\cdot(1+t)$$

almashtirish qilsak, quyidagiga ega bo'lamiz:

$$\Im = 2\cdot\int_{-1}^{1}\frac{dt}{4+(1+t)^2}.$$

$[-1,\ 1]$ oraliqda $p(t) = 1$ vazn funksiya bilan ortogonal bo'lgan ko'phad Lejandr ko'phadi deyilishi va u

$$P_n(t)=\frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n$$

formula bilan aniqlanib, ular uchun

$$(n+1)P_{n+1}(t)-(2n+1)\cdot t\cdot P_n(t)+n\cdot P_{n-1}(t)=0$$

rekurrent formula mavjudligini bilamiz. Bundan foydalanib, quyidagilarni hosil qilamiz.

$$P_0(t)=1,\ P_1(t)=t,\ P_2(t)=\frac{1}{2}\cdot(3t^2-1),\ \ P_3(t)=\frac{1}{2}\cdot(5t^3-3t),$$

$$P_4(t)=\frac{1}{8}\cdot(35t^4-30t^2+3),\qquad P_5=\frac{1}{8}\cdot(63t^5-70t^3+15t).$$

$P_5(t)=0$ ni yechib, kvadratur formula tugunlarini aniqlaymiz:

$$t_1=-\left(\frac{70+\sqrt{1120}}{126}\right)=-0,9061798,\ t_2=-\left(\frac{70-\sqrt{1120}}{126}\right)=-0,5384693,$$

$$t_3=0,\ t_4=-t_1=0,5384693,\ t_5=-t_2=0,9061798.$$

Gauss kvadratur formulasining $n=5$ dagi koeffitsiyentlarining qiymatlari quyidagicha:

$$A_1=A_5=0,2369269,$$

$$A_2=A_4=0,4786286,\quad A_3=0,568889.$$

Integral ostidagi funksiyaning tugun nuqtalardagi qiymatlari esa quyidagicha:

$$f(t_1)=0,2494511,\ f(t_2)=0,2373599,\ f(t_3)=0,2,$$

$$f(t_4)=0,1570626,\ f(t_5)=0,1310011.$$

Endi berilgan integralning taqribiy qiymatini hisoblaymiz:

$$\Im=2\sum_{i=1}^{5}A_i f(t_i)=0,7853982.$$

Berilgan integralning aniq qiymati esa

$$\Im=\frac{\pi}{4}=0,785398163\ldots$$

ga teng.

Misollar.

1) Aniq integrallarni umumlashgan o‘rta to‘g‘ri to‘rtburchaklar formulasi va umumlashgan trapetsiya formulasi bilan $n=10$ da hisoblang.

2) Aniq integrallarni Simpson kvadratur formulasi bilan $n=10$ da hisoblang.

1. 1) $\int_{0,8}^{1,6} \frac{dx}{\sqrt{2x^2+1}}$; 2) $\int_{1,2}^{2} \frac{\lg(x+2)}{x} dx$.

2. 1) $\int_{1,2}^{2,7} \frac{dx}{\sqrt{x^2+3,2}}$; 2) $\int_{1,6}^{2,4} (x+1)\sin x dx$.

3. 1) $\int_{1}^{2} \frac{dx}{\sqrt{2x^2+1,3}}$; 2) $\int_{0,2}^{1} \frac{\operatorname{tg}(x^2)}{x^2+1} dx$.

4. 1) $\int_{0,8}^{1,6} \frac{dx}{\sqrt{2x^2+1}}$; 2) $\int_{0,6}^{1,4} \frac{\cos x}{x+1} dx$.

5. 1) $\int_{0,8}^{1,4} \frac{dx}{\sqrt{2x^2+3}}$; 2) $\int_{0,6}^{1,2} \sqrt{x}\cos(x^2) dx$.

6. 1) $\int_{0,4}^{1,2} \frac{dx}{\sqrt{2+0,5x^2}}$; 2) $\int_{0,8}^{1,6} \frac{\lg(x^2+1)}{x} dx$.

7. 1) $\int_{1,4}^{2,1} \frac{dx}{\sqrt{3x^2-1}}$; 2) $\int_{0,8}^{1,2} \frac{\sin(2x)}{x^2} dx$.

8. 1) $\int_{1,2}^{2,4} \frac{dx}{\sqrt{0,5+x^2}}$; 2) $\int_{0,4}^{1,2} \frac{\cos x}{x+2} dx$.

9. 1) $\int_{0,4}^{1,2} \frac{dx}{\sqrt{3+x^2}}$; 2) $\int_{0,4}^{1,2} (2x+0,5)\sin x dx$.

10. 1) $\int_{0,6}^{1,5} \frac{dx}{\sqrt{2x^2+1}}$; 2) $\int_{0,4}^{0,8} \frac{\operatorname{tg}(x^2+0,5)}{2x^2+1} dx$.

11. 1) $\int\limits_{2}^{3,5} \frac{dx}{\sqrt{x^2-1}}$; 2) $\int\limits_{0,18}^{0,98} \frac{\sin x}{x+1} dx$.

12. 1) $\int\limits_{0,5}^{1,3} \frac{dx}{\sqrt{x^2+2}}$; 2) $\int\limits_{0,2}^{1,8} \sqrt{x+1} \cos(x^2) dx$.

13. 1) $\int\limits_{1,2}^{2,6} \frac{dx}{\sqrt{x^2+0,6}}$; 2) $\int\limits_{1,4}^{3} x^2 \lg x dx$.

14. 1) $\int\limits_{1,4}^{2,2} \frac{dx}{\sqrt{3x^2+1}}$; 2) $\int\limits_{1,4}^{2,2} \frac{\lg(x^2+2)}{x+1} dx$.

15. 1) $\int\limits_{0,8}^{1,8} \frac{dx}{\sqrt{x^2+4}}$; 2) $\int\limits_{1,4}^{1,2} \frac{\cos(x^2)}{x+1} dx$.

16. 1) $\int\limits_{1,6}^{2,2} \frac{dx}{\sqrt{x^2+2,5}}$; 2) $\int\limits_{0,8}^{1,6} (x^2+1)\sin(x-0,5) dx$.

17. 1) $\int\limits_{1,2}^{2} \frac{dx}{\sqrt{x^2+1,2}}$; 2) $\int\limits_{1,2}^{2} \frac{\lg(x^2+3)}{2x} dx$.

18. 1) $\int\limits_{3,2}^{4} \frac{dx}{\sqrt{0,5x^2+1}}$; 2) $\int\limits_{0,5}^{1,2} \frac{\operatorname{tg}(x^2)}{x+1} dx$.

19. 1) $\int\limits_{0,8}^{1,7} \frac{dx}{\sqrt{2x^2+0,3}}$; 2) $\int\limits_{1,3}^{2,1} \frac{\sin(x^2-1)}{2\sqrt{x}} dx$.

20. 1) $\int\limits_{1,2}^{2} \frac{dx}{\sqrt{0,5x^2+1,5}}$; 2) $\int\limits_{0,2}^{1} (x+1)\cos(x^2) dx$.

21. 1) $\int\limits_{1,3}^{2,1} \frac{dx}{\sqrt{3x^2-0,4}}$; 2) $\int\limits_{0,6}^{0,72} \sqrt{x+1} \operatorname{tg}(2x) dx$.

22. 1) $\int\limits_{1,4}^{2,6} \frac{dx}{\sqrt{1,5x^2+0,7}}$; 2) $\int\limits_{0,8}^{1,2} \frac{\cos x}{x^2+1} dx$.

23. 1) $\int\limits_{0,15}^{0,5} \frac{dx}{\sqrt{2x^2+1,6}}$; 2) $\int\limits_{1,2}^{2,8} \left(\frac{x}{2}+1\right) \sin\frac{x}{2} dx$.

IV BOB. ALGEBRAIK VA TRANSSENDENT TENGLAMALARNI TAQRIBIY YECHISH

4.1-§. Ildizlarni ajratish

Agar algebraik yoki transsendent tenglamaning ko'rinishi yetarlicha murakkab bo'lsa, uning ildizlarini aniq topishning har doim ham iloji bo'lavermaydi. Bundan tashqari, uning ba'zi koeffitsiyentlarining taqribiyligi ma'lum bo'lsa, ildizlarini aniq topish masalasi o'z ma'nosini yo'qotadi. Shuning uchun ildizlarni taqribiy topish metodlari va ularning aniqlik darajasini baholash muhim ahamiyatga ega.

Tenglamalarning ildizlarini taqribiy topish uchun qo'llaniladigan usullarda uning ildizlari ajratilgan, ya'ni shunday yetarli kichik oraliqlar topilganki, bu oraliqda tenglamaning bittagina ildizi joylashgan, deb faraz qilinadi. Bu oraliqning biror nuqtasini boshlang'ich yaqinlashish deb, tanlangan metod bilan berilgan aniqlikda topish mumkin. Demak, tenglama ildizlarini taqribiy topish masalasi ikki qismdan iborat:

1) ildizlarni ajratish, ya'ni shunday oraliqchalarni ko'rsatish kerakki, unda tenglamaning bitta va faqat bitta ildizi bo'lsin;

2) ildizning taqribiy qiymati – boshlang'ich yaqinlashishni berilgan aniqlikda hisoblash.

Matematik analizdan ma'lum bo'lgan quyidagi teoremalardan ildizlarni ajratishda foydalaniladi.

Teorema. Agar $f(x)$ funksiya $[a,b]$ da uzluksiz bo'lib, oraliqning oxirlarida turli ishorali qiymatlarni qabul qilsa, u holda $f(x)=0$ tenglamaning bu oraliqda hech bo'lmaganda bitta ildizi bor. Agar $f'(x)$ mavjud bo'lib, u $[a,b]$ da ishorasini saqlasa, u holda $[a,b]$ da $f(x)=0$ ning ildizi yagonadir.

Teorema. $f(x)$ funksiya $[a,b]$ da analitik bo'lib, $f(a)\cdot f(b)<0$ shart o'rinli bo'lsa, $f(x)=0$ tenglamaning $[a,b]$ da yotadigan ildizlari soni toqdir.

4.2-§ Algebraik tenglamalarning haqiqiy ildizlarini ajratish

Algebraik

$$f(x)\equiv a_0x^n+a_1x^{n-1}+\dots+a_n=0 \quad (a_0\neq 0) \tag{1}$$

tenglama ildizlarining soni va ularni ajratish masalasini ko'raylik.

Dekart teoremasi. Karraliklarining karrasi bilan hisoblaganda (1) tenglamaning musbat ildizlari soni

$$a_0, a_1, \dots, a_n$$

koeffitsiyentlar sistemasida (nolga teng koeffitsiyentlar e'tiborga olinmaydi) ishora almashtirish soniga teng yoki undan juft songa kamdir.

Gyua teoremasi. (1) tenglamaning koeffitsiyentlari haqiqiy bo'lib, uning barcha ildizlari haqiqiy bo'lsa, koeffitsiyentlar uchun

$$a_k^2>a_{k-1}\cdot a_{k+1}, \qquad i=1,2,\dots,n-1$$

tengsizliklar o'rinli.

(1) tenglamada $a_0\neq 0$, $a_n\neq 0$ deb hisoblaymiz.

Teorema 1. Agar

$$A=\max_{1\le k\le n}\left|\frac{a_k}{a_0}\right|, \quad A_1=\max_{0\le k\le n-1}\left|\frac{a_k}{a_n}\right|$$

bo'lsa, u holda (1) tenglamaning barcha ildizlari

$$r=\frac{1}{1+A_1}<|x|<1+A=R \tag{2}$$

halqa ichida yotadi.

Isboti. Faraz qilaylik $|x|>1$ bo'lsin. Modulning xossasiga ko'ra

$$|f(x)|=\left|a_0x^n\left(1+\frac{a_1}{a_0x}+\dots+\frac{a_n}{a_0x^n}\right)\right|\ge$$

$$\geq |a_0x^n|\left[1-A\left(\frac{1}{|x|}+\frac{1}{|x|^2}+\cdots\frac{1}{|x|^n}\right)\right]>$$

$$>|a_0x^n|\left[1-A\,\frac{1}{|x|-1}\right]=|a_0x^n|\left[\frac{|x|-1-A}{|x|-1}\right].$$

Agar $|x|>1+A$ deb olsak, u holda $|f(x)|>0$ tengsizlik kelib chiqadi, ya'ni x ning shunday qiymatlarida $f(x)$ ko'phad nolga aylanmaydi, demak (1) tenglamaning ildizi yo'q. (2) tengsizlikning chap tomonini ko'rsatish uchun $x=\frac{1}{y}$ deb, $f(x)=\frac{1}{y^n}g(y)$ ga ega bo'lamiz, bu yerda $g(y)=a_ny^n+a_{n-1}y^{n-1}+\cdots+a_0$. Yuqorida isbot qilinganiga ko'ra, $g(y)$ ko'phadning $y_k=\frac{1}{x_k}$ ildizlari

$$|y_k|=\frac{1}{|x_k|}<1+A_1$$

tengsizlikni qanoatlantiradi, bundan esa

$$|x_k|>\frac{1}{1+A_1}$$

kelib chiqadi.

Bu teoremadagi r va R – (1) tenglama musbat ildizlarining mos ravishda quyi va yuqori chegaralaridir. Xuddi shuningdek, $-R$ va $-r$ – manfiy ildizlarning mos ravishda quyi va yuqori chegarasi bo'ladi.

Teorema 2. (Lagranj teoremasi.) Agar (1) tenglamaning manfiy koeffitsiyentlaridan eng birinchisi (chapdan hisoblaganda) a_k bo'lib, B manfiy koeffitsiyentlarning absolut qiymatlari bo'yicha eng kattasi bo'lsa, u holda musbat ildizlarning yuqori chegarasi

$$R=1+\sqrt[k]{\frac{B}{a_0}}$$

son bilan ifodalanadi.

Isboti. Bu yerda ham $x>1$ deb olamiz. Manfiy bo'lmagan $a_1,a_2,\dots,a_{k-1}$ larni nol bilan almashtiramiz, $a_k,a_{k+1},\dots,a_n$ larni esa $-B$ ga almashtirsak $f(x)$ ko'phadning qiymati kamayishi mumkin. Shuning uchun

$$f(x)\ge a_0x^n-B\left(x^{n-k}+x^{n-k-1}+\dots+1\right)=$$

$$=\ a_0x^n-B\frac{x^{n-k+1}-1}{x-1}$$

tengsizlikka ega bo'lamiz. Bundan esa $x>1$ bo'lganda

$$f(x)\ge a_0x^n-B\ \frac{x^{n-k+1}-1}{x-1}>a_0x^n-\frac{Bx^{n-k+1}}{x-1}=$$

$$=\frac{x^{n-k+1}}{x-1}\left[a_0x^{k-1}(x-1)-B\right]>\frac{x^{n-k+1}}{x-1}\left[a_0\left(x-1\right)^k-B\right]$$

kelib chiqadi. Demak,

$$x\ge 1+\sqrt[k]{\frac{B}{a_0}}=R$$

bo'lganda $f(x)>0$ ga ega bo'lamiz, ya'ni (1) tenglamaning barcha x^+ musbat ildizlari $x^+<R$ tengsizlikni qanoatlantiradi.

Teorema 3. (Nyuton teoremasi.) Agar $x=c>0$ uchun $f^{(k)}(c)\ge 0,\ k=0,1,\dots,n$ shart o'rinli bo'lsa, $R=c$ ni (1)ning musbat ildizlari uchun yuqori chegara deb olish mumkin.

Isboti. Teylor formulasiga ko'ra

$$f(x)=\sum_{i=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k\,.$$

Shartga ko'ra, $x>c$ bo'lganda summaning har bir hadi musbatdir. Demak, (1)ning barcha x^+ musbat ildizlari $x^+\le c$ tengsizlikni qanoatlantiradi.

Agar quyidagi ko'phadlarga

$$f_1(x)=(-1)^n f(-x)=a_0x^n-a_1x^{n-1}+a_2x^{n-2}+\dots+(-1)^n a_n,$$

$$f_2(x)=x^n f\left(\frac{1}{x}\right)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_0,$$

$$f_3(x) = (-x)^n f\left(-\frac{1}{x}\right) = a_n x^n - a_{n-1}x^{n-1} + \cdots + (-1)^n a_0$$

yuqoridagi teoremalarni qo'llab, $f(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$ larning musbat ildizlarining, mos ravishda, R, R_1, R_2, R_3 yuqori chegaralari topilgan bo'lsa, u holda (1) tenglamaning hamma musbat ildizlari $\frac{1}{R_2} \le x^+ \le R$ va barcha manfiy ildizlari $-R_1 \le x^- \le -\frac{1}{R_3}$ tengsizliklarni qanoatlantiradi.

4.3-§. Tenglamalarni yechishda iteratsiya metodi

Berilgan $f(x)=0$ tenglamaning ildizlari ajratilgan bo'lsin. Iteratsiya metodini qo'llash uchun $f(x)=0$ tenglamani

$$x = \varphi(x) \tag{1}$$

kanonik ko'rinishga keltiramiz. Ildiz yotgan oraliqdan ixtiyoriy nuqta olib, izlanayotgan ildizning dastlabki yaqinlashishi deb, quyidagi ketma-ketlikni (iteratsion jarayonni) hosil qilamiz:

$$x_n = \varphi(x_{n-1}), \quad n = 1, 2, \ldots \tag{2}$$

Agar

$$\lim_{n\to\infty} x_n = \xi \tag{3}$$

mavjud va $\varphi(x)$ funksiya uzluksiz bo'lsa, (2) tenglikning har ikkala tomonida limitga o'tsak,

$$\xi = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \varphi(x_{n-1}) = \varphi\left(\lim_{n\to\infty} x_n\right) = \varphi(\xi),$$

ya'ni

$$\xi = \varphi(\xi)$$

hosil bo'ladi. Demak, ξ berilgan tenglamaning ildizi ekan va uni (2) yordamida talab qilingan aniqlikda topish mumkin. (3) limit mavjud bo'lgan holda, iteratsiya jarayoni yaqinlashuvchi deyiladi. Quyidagi teorema iteratsion jarayonning yaqinlashishini ko'rsatadi.

Teorema. Faraz qilaylik, $\varphi(x)$ funksiya va boshlang'ich yaqinlashish x_0 quyidagi shartlarni qanoatlantirsin:

1) $\varphi(x)$ funksiya

$$|x - x_0| \le \delta \tag{4}$$

oraliqda aniqlangan bo'lib, bu oraliqdan olingan ixtiyoriy ikkita x va y nuqtalar uchun $\varphi(x)$ Lipshits shartini qanoatlantirsin:

$$|\varphi(x) - \varphi(y)| \le q|x - y| \quad (0 < q < 1); \tag{5}$$

2) quyidagi tengsizliklar bajarilsin:

$$|x_0 - \varphi(x_0)| \le \eta, \; \frac{\eta}{1-q} \le \delta \; . \tag{6}$$

U holda (1) tenglama (4) oraliqda yagona ξ ildizga ega bo'lib, $\{x_n\}$ ketma-ketlik bu yechimga intiladi va intilish tezligi

$$|x_n - \xi| \le \frac{\eta}{1-q} q^n \tag{7}$$

tengsizlik bilan aniqlanadi.

Isboti. Ixtiyoriy n uchun x_n ni qurish mumkinligini va x_n (4) oraliqda yotishligi hamda

$$|x_{n+1} - x_n| \le \eta q^n \tag{8}$$

tengsizlik bajarilishini induksiya metodi bilan ko'rsatamiz.

Agar $n = 0$ bo'lsa, $x_1 = \varphi(x_0)$ bo'lgani uchun (8)dan (6) hosil bo'ladi, ya'ni

$$|x_1 - x_0| \le \eta$$

bo'ladi, bundan esa

$$|x_1 - x_0| \le \eta < \frac{\eta}{1-q} \le \delta$$

hosil bo'lib, x_1 (4) oraliqda yotishligini ko'rsatadi. Faraz qilaylik, $x_1, x_2, \ldots, x_n$ lar qurilgan bo'lib, ular (4) oraliqda yotsin va

$$|x_{k+1} - x_k| \le \eta q^k \quad (k = 0, 1, \ldots, n-1)$$

tengsizliklar bajarilsin. Induksiya shartiga ko'ra, x_n (4) da yotadi, $\varphi(x)$ (4) da aniqlangan, shuning uchun ham $x_{n+1}=\varphi(x_n)$ ni qurish mumkin. Teoremaning birinchi shartidan

$$|x_{n+1}-x_n|=|\varphi(x_{n+1})-\varphi(x_n)|\le q|x_n-x_{n-1}|$$

kelib chiqadi. x_{n-1} va x_n uchun induksiya shartiga ko'ra, $|x_n-x_{n-1}|\le\eta q^{n-1}$ o'rinli, demak $|x_{n+1}-x_n|\le\eta q^n$. Bu x_n, x_{n+1} lar uchun (8) ifoda o'rinliligini ko'rsatadi.

$$|x_{n+1}-x_0|\le|x_{n+1}-x_n|+|x_n-x_{n-1}|+\cdots+|x_1-x_0|\le$$

$$\le\eta q^n+\eta q^{n-1}+\cdots+\eta=\eta\frac{1-q^{n+1}}{1-q}<\eta\frac{1}{1-q}\le\delta$$

tengsizlik x_{n+1} (4) oraliqda yotishini ko'rsatadi.

Endi $\{x_n\}$ ketma-ketlik fundamentalligini ko'rsatamiz. (8) tengsizlikka ko'ra, ixtiyoriy p natural son uchun

$$|x_{n+p}-x_n|\le|x_{n+p}-x_{n+p-1}|+|x_{n+p-1}-x_{n+p-2}|+\cdots+|x_{n+1}-x_n|\le$$

$$\le\eta q^{n+p-1}+\eta q^{n+p-2}+\cdots+\eta q^n=\eta q^n(1+q+\cdots+q^{p-1})=$$

$$=\eta q^n\frac{1-q^p}{1-q}<\frac{\eta}{1-q}q^n$$

yoki

$$|x_{n+p}-x_n|<\frac{\eta}{1-q}q^n. \qquad (9)$$

Bu tengsizlikning o'ng tomoni p ga bog'liq emasligidan va $0<q<1$ bo'lganidan $\{x_n\}$ ketma-ketlikning fundamentalligi va uning limiti $\xi=\lim\limits_{n\to\infty}x_n$ mavjudligi kelib chiqadi. $\{x_n\}$ ketma-ketlik (4) oraliqda yotganligi uchun ξ ham shu oraliqda yotadi. (5) shartdan $\varphi(x)$ ning uzluksizligi kelib chiqadi, shuning uchun ham $x_{n+1}=\varphi(x_n)$ tenglikda limitga o'tib, ξ (1) tenglamaning ildizi ekanligini ko'ramiz.

Topilgan ildiz yagonadir. Faraz qilaylik, $\bar{\xi}$ (1) tenglamaning (4) oraliqdagi boshqa ildizi bo'lsin. (5) ga ko'ra,

$$|\bar{\xi}-\xi| = |\varphi(\bar{\xi})-\varphi(\xi)| \le q|\bar{\xi}-\xi|,$$

$0<q<1$ bo'lganligi uchun bu munosabat $\bar{\xi}=\xi$ bo'lsagina bajariladi.

(9) tengsizlikda $p\to\infty$ limitga o'tsak, (7) tengsizlik kelib chiqadi. Teorema isbot bo'ldi.

Eslatma. Faraz qilaylik, $x=\varphi(x)$ tenglamaning ildizi ξ yotgan qandaydir (a,b) oraliqda $\varphi'(x)$ ishora saqlasa va $|\varphi'(x)|\le q<1$ shart o'rinli bo'lsa, u holda agar $\varphi'(x)$ musbat bo'lsa

$$x_n=\varphi(x_{n-1}),\ \ (n=1,2,\ldots),\ \ x_0\in(a,b) \qquad (10)$$

ketma-ketlik ξ ga monoton yaqinlashadi, bordi-yu $\varphi'(x)$ manfiy bo'lsa, (10) ketma-ketlik ξ ildiz atrofida tebranib unga yaqinlashadi.

Haqiqatan ham, $0\le\varphi'(x)\le q<1$ bo'lib, $x_0<\xi$ bo'lsin. U holda

$$x_1-\xi=\varphi(x_0)-\varphi(\xi)=(x_0-\xi)\varphi'(c)<0,$$

bu yerda $c\in[x_0,\xi]$, bundan

$$|x_1-\xi|=|(x_0-\xi)\varphi'(c)|\le q|x_0-\xi|<|x_0-\xi|.$$

Demak,

$$x_0<x_1<\xi.$$

matematik induksiyaga asosan

$$x_0<x_1<x_2<\cdots<\xi$$

ga ega bo'lamiz.

Shunga o'xshash natija $x_0>\xi$ bo'lganda ham kelib chiqadi.

Endi $-1<-q\le\varphi'(x)\le 0$ holni ko'rib chiqamiz.

Faraz qilaylik, $x_0<\xi$ bo'lib, $x_1=\varphi(x_0)\in(a,b)$ bo'lsin, u holda

$$x_1-\xi=\varphi(x_0)-\varphi(\xi)=(x_0-\xi)\varphi'(c)>0,$$

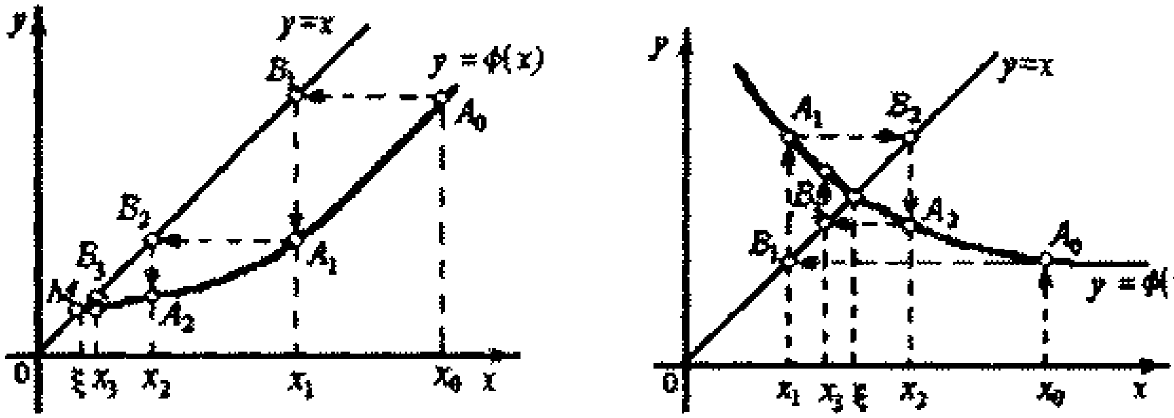
bo'ladi, bundan $x_1>\xi$ va $|x_1-\xi|<|x_0-\xi|$ ligi kelib chiqadi.

Shu mulohazalarni $x_1, x_2, \ldots,$ yaqinlashishlar uchun qaytarsak,

$$x_0 < x_2 < x_4 < \cdots < \xi < \cdots < x_3 < x_1$$

hosil bo'ladi, ya'ni ketma-ket yaqinlashishlar ξ atrofida tebranib, unga yaqinlashadi.

Bu ikkala holning geometrik talqinini quyidagi rasmda izohlaymiz.



4.4-§. Nyuton metodi

Faraz qilaylik,

$$f(x) = 0 \qquad (1)$$

tenglamaning $[a,b]$ oraliqdagi yagona ildizi ξ bo'lsin. $f'(x)$ va $f''(x)$ lar noldan farqli bo'lib, $[a,b]$ da ishora saqlasin. $x_n \in [a,b]$ bo'lib, ξ ning taqribiy qiymati bo'lsin, ya'ni

$$\xi = x_n + \varepsilon_n, \qquad (2)$$

ε_n xatolik va uni kichik miqdor bo'lsin deb hisoblaymiz. Teylor formulasiga asosan

$$0 = f(x_n + \varepsilon_n) \cong f(x_n) + \varepsilon_n \cdot f'(x_n)$$

taqribiy tenglikka ega bo'lamiz. Bundan $\varepsilon_n \cong -\dfrac{f(x_n)}{f'(x_n)}$ bo'lganligi uchun (2) dan ildizning navbatdagi taqribiy qiymatiga ega bo'lamiz:

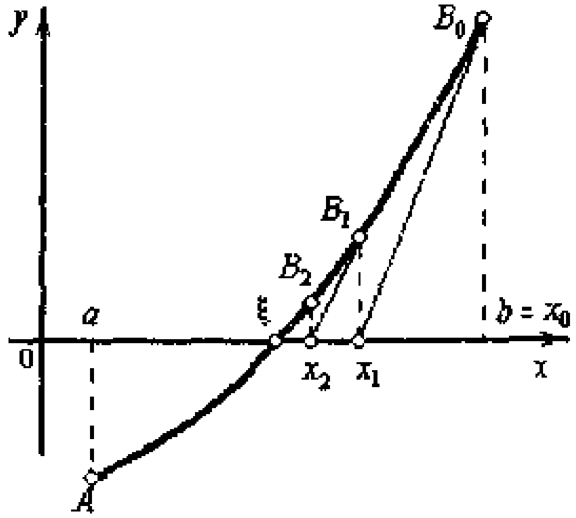
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0,1,2,...). \qquad (3)$$

(3) ketma-ketlikni qurishda boshlang'ich yaqinlashish $x_0 \in [a,b]$ bo'lib, $f(x_0) \cdot f''(x_0) > 0$ shartni qanoatlantirishi maqsadga muvofiqdir.

(3) ketma-ketlikning geometrik talqini quyidagidan iborat:

x_{n+1} ning qiymati $y = f(x)$ funksiya grafigining $(x_n, f(x_n))$ nuqtasiga o'tkazilgan urinmaning OX o'qi bilan kesishgan nuqtasining abssissasiga tengdir. Shuning uchun ham Nyuton metodi urinmalar metodi deb ham ataladi.

(3) formula bilan topilgan ketma-ketlik (1) tenglamaning ildizi ξ ga boshlang'ich yaqinlashish x_0 o'ng tomondan monoton yaqinlashishini quyidagi rasmdan ko'rish mumkin.



Nyuton metodining yaqinlashish tezligini quyidagicha baholash mumkin. Teylor formulasidan

$$0 = f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{1}{2} f''(\eta)(\xi - x_n)^2,$$

bu yerda η, ξ va x_n oralig'ida joylashgan.

Bundan, $\frac{f(x_n)}{f'(x_n)} = x_n - \xi - \frac{1}{2} \frac{f''(\eta)}{f'(x_n)} (\xi - x_n)^2$.

Demak,

$$x_{n+1}-\xi = x_n-\xi-\frac{f(x_n)}{f'(x_n)}=\frac{1}{2}\frac{f''(\eta)}{f'(x_n)}(\xi-x_n)^2 .$$

Agar $m=\min\limits_{[a,b]}|f'(x)|$, $M=\max\limits_{[a,b]}|f''(x)|$, $[a,b]$ x_0 va ξ ni o'z ichiga olgan hamda $f'(x)$, $f''(x)$ ishorasini o'zgartirmaydigan oraliq bo'lsa, u holda

$$|x_{n+1}-\xi|\le\frac{M}{2m}|x_n-\xi|^2$$

ga ega bo'lamiz. Bu Nyuton metodining yaqinlashish tezligini ko'rsatadi.

Eslatma. Agar iteratsiya usulida

$$\varphi(x)=x-\frac{f(x)}{f'(x)}$$

deb olsak, Nyuton metodi hosil bo'ladi.

Agar $[a,b]$ da $f'(x)$ kam o'zgarsa, u holda (3) formulada $f'(x_n)\cong f'(x_0)$ deyish mumkin va natijada Nyuton metodining modifikatsiyasi

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_0)}\quad (n=0,1,2,...) \tag{4}$$

hosil bo'ladi. (4) formula bilan hisoblash jarayoni o'tkazilsa, har bir qadamda $f'(x)$ ning qiymatini hisoblanmaslik qulaylik tug'diradi. Bu qulaylik, ayniqsa $f'(x)$ murakkab bo'lganda yaqqol seziladi.

4.5-§. Yuqori tartibli iteratsion metod qurishda Chebishev metodi

Faraz qilaylik, $f(x)=0$ tenglamani $[a,b]$ da yagona ildizi $x=\xi$ mavjud bo'lsin va $f(x)$ funksiya hamda uning yetarlicha yuqori tartibli hosilalari uzluksiz bo'lsin. Bundan tashqari, $x\in[a,b]$ da $f'(x)\ne 0$ bo'lsin. U holda $f'(x)$ bu oraliqda ishora saqlaydi va $f(x)$

monoton funksiya bo'lib, $x=g(y)$ teskari funksiyaga ega bo'ladi. Teskari funksiya $g(y)$ $f(x)$ning o'zgarish sohasi $[c,d]$ da aniqlangan bo'lib, $f(x)$ qancha uzluksiz hosilaga ega bo'lsa, u ham shuncha uzluksiz hosilaga ega bo'ladi. Teskari funksiya ta'rifiga ko'ra,

$$x=g(y) \quad (x\in[a,b]), \quad y=f(g(y)) \ (y\in[c,d])$$

Demak,

$$\alpha=g(0). \qquad (1)$$

Agar $y\in[c,d]$ bo'lsa, u holda Teylor formulasidan

$$\alpha=g(0)=g(y-y)=g(y)+\sum_{k=1}^{p-1}(-1)^k\frac{g^{(k)}(y)}{k!}y^k+(-1)^p\frac{g^{(p)}(\eta)}{p!}y^p, \qquad (2)$$

bu yerda $\eta\in[0,y]$. (2) da $y=f(x)$ va $g(y)=x$ ni nazarda tutib,

$$\alpha=x+\sum_{k=1}^{p-1}(-1)^k\frac{g^{(k)}(f(x))}{k!}f^k(x)+(-1)^p\frac{g^{(p)}(f(x))}{p!}f^p(x)$$

ni hosil qilamiz. Agar

$$\varphi_p(x)=x+\sum_{k=1}^{p-1}(-1)^k\frac{g^{(k)}(f(x))}{k!}f^k(x)$$

deb belgilasak, u holda

$$x=\varphi_p(x)$$

tenglama uchun $x=\xi$ yechim bo'ladi, chunki

$$\varphi_p(\xi)=\xi+\sum_{k=1}^{p-1}(-1)^k\frac{g^{(k)}(f(\xi))}{k!}f^k(\xi)=\xi.$$

Bundan

$$\varphi_p^{(j)}(\xi)=0, \quad j=1,2,\ldots,p-1$$

bo'lganligi sababli

$$x_{n+1}=\varphi_p(x_n), \quad n=0,1,2,\ldots; \ x_0\in[a,b] \qquad (4)$$

iteratsion jarayon p-tartibli deyiladi. Agar x_0 ξ ga yaqin bo'lsa, u holda (4) bilan aniqlangan $\{x_n\}$ ketma-ketlik ξ ga yaqinlashadi. Haqiqatan ham, $\varphi_p'(\xi)=0$ bo'lganligi uchun ξ ning shunday atrofi topiladiki, u yerda $|\varphi'_p(x)| \le q < 1$ bo'ladi. Bundan x_0 ξ ga yetarlicha yaqin bo'lsa, $\{x_n\}$ iteratsion ketma-ketlikning yaqinlashishi kelib chiqadi.

Endi $x = g(f(x))$ dan ketma-ket hosila olamiz:

$$g'(f(x))f'(x) = 1,$$

$$g''(f(x))\cdot f'^2(x) + g'(f(x))\cdot f''(x) = 0,$$

$$g'''(f(x))\cdot f'^3(x) + 3g''(f(x))\cdot f'(x)\cdot f''(x) + g'(f(x))\cdot f'''(x) = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Bundan ketma-ket $g'(f(x)), g''(f(x)), \ldots, g^{(p-1)}(f(x))$ larni va shu bilan birga $\varphi_p(x)$ ni aniqlaymiz. (4) iteratsion jarayonni p ning bir nechta konkret qiymatlarida ko'rinishini keltiramiz:

$$p = 2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$p = 3$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)\cdot f^2(x_n)}{2[f'(x_n)]^3},$$

$$p = 4$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)\cdot f^2(x_n)}{2[f'(x_n)]^3} - \frac{f^3(x_n)}{12}\cdot\frac{3[f''(x_n)]^2 - f'(x_n)\cdot f'''(x_n)}{[f'(x_n)]^5}.$$

Bular mos ravishda 2-, 3- va 4-tartibli iteratsion jarayonlardir.

Bobga tegishli tayanch so'zlar: ildizlarni ajratish, ildizlar chegarasi, iteratsion jarayon, yaqinlashish tezligi, iteratsion jarayon tartibi, yuqori tartibli iteratsion metod.

Savollar va topshiriqlar

1. Algebraik yoki transsendent tenglama ildizlarini taqribiy topish masalasi necha qismdan iborat?

2. $[a,b]$ da $f(x)=0$ tenglamaning ildizi yagona bo'lishligi uchun qanday shart bajarilishi kerak?

3. Algebraik tenglamaning musbat ildizlari soni haqidagi Dekart teoremasini ayting.

4. Gyua teoremasini ayting.

5. Algebraik tenglamaning barcha ildizlari yotadigan sohani ayting.

6. Algebraik tenglamaning musbat ildizlarining yuqori chegarasi haqidagi Lagranj teoremasi.

7. Algebraik tenglamaning musbat ildizlarining yuqori chegarasi haqidagi Nyuton teoremasi.

8. Algebraik tenglama manfiy ildizlarining quyi va yuqori chegaralari qanday aniqlanadi?

9. Iteratsiya metodining mohiyatini ayting.

10. Qanday shartlar o'rinli bo'lsa, $\{x_n\}$ ketma-ketlik aniq ildiz ξ ga monoton yaqinlashadi (geometrik talqinini ko'rsating)?

11. Qanday shartlar o'rinli bo'lsa, $\{x_n\}$ ketma-ketlik aniq ξ ildiz atrofida tebranib yaqinlashadi (geometrik talqinini ko'rsating)?

12. Iteratsiya metodining yaqinlashish tezligi.

13. Nyuton metodi.

14. Nyuton metodida boshlang'ich yaqinlashish x_0 qanday aniqlanadi?

15. Nyuton metodining yaqinlashish tezligi.

16. Qanday shart o'rinli bo'lsa Nyuton metodining modifikatsiyasini ishlatgan ma'qul?

17. Chebishev metodi.

18. Uchinchi tartibli iteratsion jarayonni hosil qiling.

Misol 1. $f(x) \equiv x^4 - 5x^2 + 8x - 8 = 0$ tenglamaning haqiqiy ildizlari chegarasini toping. Haqiqiy va kompleks ildizlar sonini aniqlang.

Yechish. Teorema 1 ga asosan, $a_0 = 1$, $A = 8$. Demak, $R = 1 + 8 = 9$, ya'ni tenglamaning haqiqiy ildizlari $(-9,9)$ oraliqda joylashgan.

Lagranj teoremasini qo'llasak, $a_0 = 1$, $k = 2$, $B = 8$. Musbat ildizlar uchun yuqori chegara

$$R = 1 + \sqrt{\frac{8}{1}} = 1 + 2\sqrt{2} < 4$$

bo'ladi. Berilgan tenglamada $x = -x$ almashtirish qilsak, $f_1(x) = f(-x) \equiv x^4 - 5x^2 - 8x - 8 = 0$ tenglama hosil bo'ladi. Lagranj teoremasiga ko'ra bu tenglamaning musbat ildizlarining yuqori chegarasini aniqlaymiz: $a_0 = 1$, $k = 2$, $B = 8$ bo'lib, $R_1 < 4$ ligi kelib chiqadi, ya'ni ildizlar $(-4,4)$ oraliqda yotadi.

Endi Nyuton teoremasini qo'llab ko'ramiz:

$$f(x) = x^4 - 5x^2 + 8x - 8,$$
$$f'(x) = 4x^3 - 10x + 8,$$
$$f''(x) = 12x^2 - 10,$$
$$f'''(x) = 24x,$$
$$f^{IV}(x) = 24.$$

Bundan ko'rinib turibdiki, $x > 2$ uchun $f^{(k)}(2) > 0$, $k = 0,1,2,3,4$, demak, $c = 2$ musbat ildizlarning yuqori chegarasi ekan. Endi $f_1(x) = 0$ tenglama uchun ildizlarning yuqori chegarasini topamiz:

$$f_1(x) = x^4 - 5x^2 - 8x - 8,$$
$$f_1'(x) = 4x^3 - 10x - 8,$$
$$f_1''(x) = 12x^2 - 10,$$
$$f_1'''(x) = 24x,$$
$$f_1^{IV}(x) = 24.$$

$f_1^{(k)}(x) > 0,\ k = \overline{0,4}$ shartlar $x > 3$ uchun bajarilishini anglash qiyin emas. Demak, berilgan tenglamaning barcha haqiqiy ildizlari $(-3;2)$ oraliqda yotar ekan.

Yuqorida berilgan tenglamaning kompleks ildizlari bor-yo'qligini Gyua teoremasidan aniqlash mumkin.

$f(x)$ ning koeffitsiyentlari

$$1, 0, 5, -8, 8$$

sonlardan tuzilgan sistemada

$$a_k^2 > a_{k-1} \cdot a_{k+1},\quad k = 1,2,3$$

shartlar o'rin bo'lsa, barcha ildizlar haqiqiy, aks holda kompleks ildizlar mavjud bo'ladi. Berilgan misolda $k = 1$ uchun ko'rsatilgan shart bajarilmaydi. Demak, tenglama 2 ta haqiqiy, 2 ta kompleks ildizga ega.

Misol 2. $f(x) \equiv x^3 - 20x + 12 = 0$ tenglama ildizlarini ajrating va ajratilgan ildizlarni iteratsiya, Nyuton metodlari bilan aniqlaydigan jarayonni ko'rsating.

Yechish. Ildizlar chegarasini Lagranj teoremasiga ko'ra aniqlaymiz: $a_0 = 1,\ \ k = 2,\ \ a_2 = -20,\ \ R^+ < 1 + \sqrt{\frac{20}{1}} < 5{,}5$, tenglamada $x = -x$ almashtirish o'tkazib, hosil bo'lgan tenglamaga Lagranj teoremasini qo'llasak, $a_0 = 1,\ k = 2,\ B = 20,\ R^+ < 1 + \sqrt{\frac{20}{1}} < 5{,}5$ bo'ladi. Demak, berilgan tenglamaning haqiqiy ildizlari $(-5{,}5\,;\ 5{,}5)$ oraliqda yotadi. Quyidagini aniqlaymiz:

x	–5,5	–5	–4	0	1	4	5	5,5
sign $f(x)$	–	–	+	+	–	–	+	+

Ya'ni berilgan tenglama 3 ta haqiqiy ildizga ega bo'lib, ular $(-5;-4)$, $(0;1)$ va $(4{,}5)$ oraliqlarda joylashgan.

Iteratsiya metodi. A) $(-5;-4)$ oraliqdagi ildiz uchun berilgan tenglamani $x=\sqrt[3]{20x-12}=\varphi(x)$ ko‘rinishga keltiramiz, chunki

$$\max_{[-5,-4]}|\varphi'(x)|=\max_{[-5,-4]}\left|\frac{20}{3\sqrt[3]{(20x-12)^2}}\right|<\frac{20}{3\sqrt[3]{92^2}}<\frac{20}{3\cdot 4{,}5^2}=\frac{80}{243}<\frac{1}{3}$$

bo‘ladi. Hisoblash jarayoni

$$x_{n+1}=\sqrt[3]{20x_n-12},\quad n=0,1,\ldots$$

formula bilan tashkil etiladi, bu yerda x_0 ixtiyoriy bo‘lib, $[-5,-4]$ oraliqqa tegishli.

B) $(0;1)$ oraliqdagi ildiz uchun

$$x=\frac{x^3+12}{20}=\varphi(x)$$

deb olamiz, chunki

$$\max_{[0,1]}|\varphi'(x)|=\max_{[0,1]}\left|\frac{3x^2}{20}\right|\le\frac{3}{20}<1$$

bo‘ladi. Hisoblash jarayoni

$$x_{n+1}=\frac{x_n^3+12}{20},\quad n=0,1,2,\ldots$$

formula bilan tashkil etiladi va x_0 sifatida $[0;1]$ ga tegishli ixtiyoriy qiymatni olish mumkin.

D) $(4,5)$ oraliqdagi ildiz uchun ham hisoblash jarayoni

$$x_{n+1}=\sqrt[3]{20x_n-12},\quad n=0,1,\ldots$$

formula bilan amalga oshiriladi, x_0 ixtiyoriy bo‘lib, $[4,\ 5]$ oraliqqa tegishli.

Nyuton metodi: $f'(x)=3\cdot x^2-20$, $f''(x)=6\cdot x$

A) $(-5,-4)$ oraliqdagi ildiz uchun $f(x_0)\cdot f''(x_0)>0$ shartni tekshiramiz:

$$f(-5)<0,\ f(-4)>0,\ f''(-5)<0,\ f''(-4)<0.$$

$f(-5)\cdot f''(-5)>0$ bo‘lganligi uchun $x_0=-5$ bo‘ladi.

Xuddi shuningdek, B) (0;1) oraliq uchun $x_0=0$, D) (4;5) oraliq uchun $x_0=5$ bo'lishligini aniqlaymiz va uch holda ham

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},\quad n=0,1,\dots$$

iteratsion jarayon bilan ildizning taqribiy qiymati berilgan aniqlikda hisoblanadi.

Misollar.

Tenglamalar ildizlarini ajrating va ajratilgan ildizlarni iteratsiya, Nyuton va vatarlar metodlari bilan aniqlang ($\varepsilon=10^{-3}$ aniqlikda).

1. $x^2-5\sin x=0.$
2. $\sin x-1/x=0.$
3. $\lg x-\cos x=0.$
4. $4x-5\ln x=5.$
5. $e^x-10x=0.$
6. $0{,}25x^3+x-2=0,\ x\in[0,2].$
7. $3x-4\ln x-5=0,\ x\in[2,4].$
8. $\cos\left(\frac{2}{x}\right)-2\sin\left(\frac{1}{x}\right)+\frac{1}{x}=0,\ x\in[1,2].$
9. $x-\frac{1}{3+\sin(3{,}6x)}=0,\ x\in[0,1].$
10. $e^{x-1}-x^3-x=0,\ x\in[0,1].$
11. $\arccos x-\sqrt{1-0{,}3x^3}=0,\ x\in[0,1].$
12. $\sqrt{1-0{,}4x^2}-\arcsin x=0,\ x\in[0,1]$
13. $3x-14+e^x-e^{-x}=0,\ x\in[1,3]$
14. $\sqrt{2x^2+1{,}2}-\cos x-1=0,\ x\in[0,1].$

15. $0,1x^2 - x\ln x = 0,\ \ x \in [1, 2]$.

16. $e^x - e^{-x} - 2 = 0,\ \ x \in [0, 1]$.

17. $\sqrt{1-x} - \mathrm{tg}\, x = 0,\ \ x \in [0, 1]$.

18. $1 - x + \sin x - \ln(1 + x) = 0,\ \ x \in [0, 2]$.

19. $x^3 - 2x^2 + x - 3 = 0$.

20. $x^3 - 2x^2 + 3x - 5 = 0$.

21. $x^4 - 5x^3 + 2x^2 - 10x + 1 = 0$.

22. $x^3 + x = 1000$.

23. $x^4 - 2x^3 + x^2 - 12x + 20 = 0$.

24. $x^4 + 6x^3 + x^2 - 4x - 60 = 0$.

25. $x^4 - 14x^2 - 40x - 75 = 0$.

26. $x^4 - x^3 + x^2 - 11x + 10 = 0$.

27. $x^4 - x^3 - 29x^2 - 71x - 140 = 0$.

28. $x^4 + 7x^3 + 9x^2 + 13x - 30 = 0$.

29. $x^4 + 3x^3 - 23x^2 - 55x - 150 = 0$.

30. $x^4 - 6x^3 + 4x^2 + 10x + 75 = 0$.

V BOB. CHIZIQLI ALGEBRAIK TENGLAMALAR SISTEMASINI YECHISH

5.1-§. Metodlar klassifikatsiyasi

Ma'lumki, ham tatbiqiy, ham nazariy matematikaning ko'pgina masalarini hal qilishda chiziqli algebraik tenglamalar sistemasini yechishga to'g'ri keladi. Masalan, funksiyani interpolyatsiyalash yoki o'rta kvadratik ma'noda yaqinlashtirish masalalari chiziqli algebraik tenglamalar sistemasini yechishga keltiriladi.

Chiziqli algebraik tenglamalar sistemasini hosil qilishning asosiy manbalaridan biri uzluksiz funksional tenglamalarni chekli-ayirmali tenglamalar sistemasi bilan yaqinlashtirishdir.

Chiziqli algebraik tenglamalar sistemasini yechishda foydalaniladigan metodlarni ikki guruhga ajratish mumkin: aniq va iteratsion metodlar.

Aniq metod deganda shunday metod tushuniladiki, uning yordamida chekli miqdordagi arifmetik amallarni aniq bajarish natijasida masalaning yechimi topiladi. Ko'pincha, bu ikki bosqichdan iborat bo'ladi. Birinchi bosqichda berilgan tenglamalar sistemasi u yoki bu ma'noda soddaroq tenglamalar sistemasiga almashtiriladi. Ikkinchi bosqichda almashtirilgan soddaroq tenglamalar sistemasi yechilib, noma'lumlarning qiymati aniqlanadi.

Iteratsion metodlar chiziqli algebraik tenglamalar sistemasining yechimi ketma-ket yaqinlashishlarning limiti ko'rinishida izohlanishi bilan xarakterlanadi.

Iteratsiya metodlarini qo'llaganda faqat ularning yaqinlashishlarigina emas, balki yaqinlashishlarning tezligi ham katta ahamiyatga ega.

5.2-§. Gauss metodi

Bu metod bir necha hisoblash sxemalariga ega. Biz Gaussning kompakt sxemasigina bilan tanishamiz.

Quyidagi sistema berilgan bo'lsin:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = a_{1,n+1} \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = a_{2,n+1} \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = a_{n,n+1} \end{cases} \qquad (1)$$

Faraz qilaylik, $a_{11} \neq 0$ (yetakchi element) bo'lsin deb, sistemaning birinchi tenglamasidan

$$x_1 + b_{12}^{(1)}x_2 + b_{13}^{(1)}x_3 + \dots + b_{1n}^{(1)}x_n = b_{1,n+1}^{(1)} \qquad (2)$$

ni hosil qilamiz, bu yerda

$$b_{1j}^{(1)} = a_{ij}/a_{11}, (j \geq 2).$$

(2) dan foydalanib, (1) sistemaning qolgan tenglamalaridan x_1 noma'lumni yo'qotish mumkin, ya'ni (2)ni ketma-ket $a_{21}, a_{31}, \dots, a_{n1}$ larga ko'paytirib, mos ravishda, ikkinchi, uchinchi va h.k. tenglamalaridan ayirsak, natijada quyidagi sistema hosil bo'ladi:

$$\begin{cases} a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + \dots + a_{2n}^{(1)}x_n = a_{2,n+1}^{(1)}, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{n2}^{(1)}x_2 + a_{n3}^{(1)}x_3 + \dots + a_{nn}^{(1)}x_n = a_{n,n+1}^{(1)} \end{cases} \qquad (3)$$

bu yerda $\qquad a_{ij}^{(1)} = a_{ij} - a_{i1}b_{1j}^{(1)} \quad (i, j \geq 2).$

(3) tenglamalar sistemasida $a_{22}^{(1)} \neq 0$ deb, yuqoridagidek jarayonni bajarsak,

$$\begin{cases} a_{33}^{(2)}x_3 + \dots + a_{3n}^{(2)}x_n = a_{3,n+1}^{(2)}, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{n3}^{(2)}x_3 + \dots + a_{nn}^{(2)}x_n = a_{n,n+1}^{(2)} \end{cases}$$

sistemaga kelamiz, bu yerda

$$a_{ij}^{(2)} = a_{ij}^{(1)} - a_{i2}^{(1)} b_{2j}^{(2)} \quad (i, j \geq 3).$$

Shu jarayonni n marta bajarish mumkin bo'lgan bo'lsa, (1) tenglamalar sistemasi

$$\begin{cases} x_1 + b_{12}^{(1)} x_2 + b_{13}^{(1)} x_3 + \cdots + b_{1n}^{(1)} x_n = b_{1,n+1}^{(1)}, \\ \qquad x_2 + b_{23}^{(2)} x_3 + \cdots + b_{2n}^{(2)} x_n = b_{2,n+1}^{(2)}, \\ \qquad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \qquad \qquad \qquad \qquad x_n = b_{n,n+1}^{(n)} \end{cases} \tag{4}$$

ko'rinishga keladi. Bu tenglamalar sistemasidan ketma-ket $x_n, x_{n-1}, \ldots, x_1$ lar aniqlanadi. (1) dan qadamma-qadam (4) ko'rinishga kelish Gauss metodining to'g'ri yo'li, (4) dan ketma-ket $x_n, x_{n-1}, \cdots, x_1$ larni aniqlash Gauss metodining teskari yo'li deyiladi.

Faraz qilaylik, Gauss metodida to'g'ri yo'lning $m \, (m < n)$ ta qadami bajarilishi mumkin bo'lgan bo'lsin, u holda quyidagiga ega bo'lamiz:

$$\begin{cases} x_m + b_{m,m+1}^{(m)} x_{m+1} + \cdots + b_{mn}^{(m)} x_n = b_{m,n+1}^{(m)}, \\ \quad a_{m+1,m+1}^{(m)} x_{m+1} + \cdots + a_{m+1,n}^{(m)} x_n = a_{m+1,n+1}^{(m)}, \\ \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \quad a_{n,m+1}^{(m)} x_{m+1} + \cdots + a_{n,n}^{(m)} x_n = a_{n,n+1}^{(m)} \end{cases} \tag{5}$$

bu yerda

$$b_{mj}^{(m)} = \frac{a_{mj}^{(m)}}{a_{mm}^{(m)}}, \quad a_{ij}^{(m)} = a_{ij}^{(m-1)} - a_{im}^{(m-1)} b_{mj}^{(m)}, \quad (i, j \geq m+1).$$

Gauss metodida bajarilishi mumkin bo'lgan qadamlarning soni m ga teng bo'lgan bo'lsa, bu shuni anglatadiki, (5) sistemaning ikkinchi tenglamasidan boshlab yetakchi elementni ajratish mumkin emas, chunki barcha $a_{ij}^{(m)} (i, j = m+1, \cdots, n)$ lar nolga teng.

Agar (5) da $a_{i,n+1}^{(m)} = 0 \ (i = m+1,\cdots,n)$ nolga teng bo'lsa, (5) bitta tenglamadan iborat bo'ladi.

Endi barcha qadamdagi birinchi tenglamalarni birlashtirib,

$$\begin{cases} x_1 + b_{12}^{(1)}x_2 + b_{13}^{(1)}x_3 + \cdots + b_{1n}^{(1)}x_n = b_{1,n+1}^{(1)} \\ x_2 + b_{23}^{(2)}x_3 + \cdots + b_{2n}^{(2)}x_n = b_{2,n+1}^{(2)} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ x_m + b_{m,m+1}^{(m)}x_{m+1} + \cdots + b_{mn}^{(m)}x_n = b_{m,n+1}^{(m)} \end{cases}$$

sistemani hosil qilamiz. Bundan $x_1, x_2, \ldots, x_m$ larni $x_{m+1}, x_{m+2}, \ldots, x_n$ lar orqali ifodalab olishimiz mumkin. Bu holda (1) cheksiz ko'p yechimga ega bo'ladi. Agar (5) da $a_{i,m+1}^{(m)}$ $(m+1 \le i \le n)$ larning hech bo'lmaganda birortasi noldan farqli bo'lsa, u holda (1) yechimga ega emas bo'ladi.

5.3-§. Kvadrat ildizlar metodi

Faraz qilaylik,

$$Ax = b \tag{1}$$

chiziqli algebraik tenglamalar sistemasida A simmetrik matritsa bo'lsin, ya'ni $A' = \left[a_{ji}\right] = A$. Soddalik uchun kvadrat ildizlar metodini shu holda bayon etamiz.

A matritsa simmetrik bo'lgani uchun uni

$$A = T'T \tag{2}$$

ko'rinishida yozish mumkin, bu yerda

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & t_{nn} \end{bmatrix}, \quad T' = \begin{bmatrix} t_{11} & 0 & \ldots & 0 \\ t_{12} & t_{22} & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ t_{1n} & t_{2n} & \cdots & t_{nn} \end{bmatrix}.$$

T' ni T ga ko'paytirib, t_{ij} larni aniqlash uchun quyidagi tenglamalarga ega bo'lamiz:

$$\begin{cases} t_{1i}t_{1j} + t_{2i}t_{2j} + \cdots + t_{ii}t_{jj} = a_{ij} \ , (i < j) \\ t_{11}^2 + t_{22}^2 + \cdots + t_{ii}^2 = a_{ii} \end{cases}$$

Bundan ketma-ket

$$\begin{cases} t_{11} = \sqrt{a_{11}} \ , t_{1j} = \dfrac{a_{1j}}{t_{11}}, \ (j \geq 2), \\ t_{ii} = \sqrt{a_{ii} - \sum\limits_{k=1}^{i-1} t_{ki}^2}, \quad (1 < i \leq n), \\ t_{ij} = \dfrac{a_{ij} - \sum\limits_{k=1}^{i-1} t_{ki}t_{kj}}{t_{ii}}, \ (i < j), \\ t_{ij} = 0 \ \text{ agar } \ i > j \text{ bo‘lsa} \end{cases}$$

larni aniqlaymiz.

(1) sistema yagona yechimga ega bo‘ladi, agar $t_{ii} \neq 0$, $i = 1, 2, \cdots, n$ bo‘lsa, chunki

$$\det A = \det T' \cdot \det T = (\det T)^2 = (t_{11}t_{22} \cdots t_{nn})^2 \neq 0 ,$$

(2) ni e’tiborga olsak, (1) sistema ikkita $T'y = b$ va $Tx = y$ tenglamalar sistemasiga ekvivalent bo‘ladi. Bu tenglamalar sistemasini ochib yozamiz:

$$\begin{cases} t_{11}y_1 = b_1 \\ t_{12}y_1 + t_{22}y = b_2 \\ \dots \ \dots \ \dots \ \dots \ \dots \\ t_{1n}y_1 + t_{2n}y_2 + \cdots + t_{nn}y_n = b_n \end{cases} \tag{4}$$

va

$$\begin{cases} t_{11}x_1 + t_{12}x_2 + \cdots + t_{1n}x_n = y_1 \\ t_{22}x_2 + \cdots + t_{2n}x_n = y_2 \\ \dots \ \dots \ \dots \ \dots \ \dots \\ t_{nn}x_n = y_n \end{cases} \tag{5}$$

(4) dan ketma-ket,

$$\begin{cases} y_1 = \dfrac{b_1}{t_{11}}, \\ y_i = \dfrac{b_i - \sum\limits_{k=1}^{i-1} t_{ki} y_k}{t_{ii}} \quad (i > 1) \end{cases} \tag{6}$$

larni topamiz. Topilganlardan foydalanib (5) dan ketma-ket

$$\begin{cases} x_n = \dfrac{y_n}{t_{nn}}, \\ x_i = \dfrac{y_i - \sum\limits_{k=i+1}^{n} t_{ik} x_k}{t_{ii}} \quad (i < n) \end{cases}$$

larni aniqlaymiz.

Shuni eslatib o‘tish lozimki, agar qandaydir s-satr uchun $t_{ss}^2 < 0$ bo‘lsa, mos ravishda t_{sj} elementlar mavhum bo‘ladi.

Metodni bu holda ham qo‘llash mumkin.

5.4-§. Haydash usuli

Chiziqli algebraik tenglamalar sistemasining matritsasi uch diagonalli bo‘lgan holni qaraymiz:

$$\begin{cases} b_1 x_1 + c_1 x_2 = d_1 \\ a_2 x_1 + b_2 x_2 + c_2 x_2 = d_2 \\ a_3 x_2 + b_3 x_3 + c_3 x_4 = d_3 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n-1} x_{n-2} + b_{n-1} x_{n-1} + c_{n-1} x_n = d_{n-1}, \\ a_n x_n + b_n x_n = d_n \end{cases} \tag{1}$$

Bu tenglamalar sistemasining kengaytirilgan matritsasini yozaylik.

$$\left(\begin{array}{cccccccc|c} b_1 & c_2 & 0 & 0 & \dots & 0 & 0 & 0 & d_1 \\ a_2 & b_2 & c_2 & 0 & \dots & 0 & 0 & 0 & d_2 \\ 0 & a_3 & b_3 & c_3 & \dots & 0 & 0 & 0 & d_3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & b_{n-1} & c_{n-1} & d_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & a_n & b_n & d_n \end{array}\right) \qquad (2)$$

(1) tenglamalar sistemasini quyidagicha kompakt shaklda yozamiz:

$$a_k x_{k-1} + b_k x_k + c_k x_{k+1} = d_k, a_1 = 0, c_n = 0 \quad (k = 1, 2, \dots, n). \qquad (3)$$

(1) ni Gauss metodi bilan yechamiz va metod to‘g‘ri yo‘lining barcha qadami bajarilgan bo‘lsin, u holda (2) quyidagi ko‘rinishga kelgan bo‘ladi:

$$\left(\begin{array}{cccccccc|c} 1 & p_1 & 0 & 0 & \dots & 0 & 0 & 0 & q_1 \\ 0 & 1 & p_2 & 0 & \dots & 0 & 0 & 0 & q_2 \\ 0 & 0 & 1 & p_3 & \dots & 0 & 0 & 0 & q_3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & p_{n-1} & q_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & q_n \end{array}\right)$$

Bu yerdagi p_k va q_k lar haydash koeffitsiyentlari deyiladi va ularni aniqlaydigan formulani chiqaramiz.

Gauss usulining teskari yo‘li yordamida noma’lumlarning qiymatlarini topish mumkin. Ular quyidagi rekurrent formula bilan aniqlanadi:

$$x_n = q_n, \quad x_k = q_k - p_k x_{k+1}, \quad k = n-1, n-2, \dots, 1. \qquad (4)$$

(1) va (4) dan foydalanib, x_k ni x_{k+1} orqali ifodalaymiz:

$$a_k\left(q_{k-1} - p_{k-1}x_k\right) + b_k x_k + c_k x_{k+1} = d_k,$$

bundan

$$x_k = \frac{d_k - a_k q_{k-1}}{b_k - a_k p_{k-1}} - \frac{c_k}{b_k - a_k p_{k-1}} x_{k+1}. \quad (5)$$

(1) ning birinchi tenglamasidan p_1, q_1 larning qiymatini aniqlaymiz. Qolgan haydash koeffitsiyentlari (4) va (5) ni taqqoslash natijasida hosil bo'ladigan rekurrent formula yordamida aniqlanadi:

$$p_1 = \frac{c_2}{b_1}, \quad q_1 = \frac{d_1}{b_1}, \quad p_k = \frac{c_k}{b_k - a_k p_{k-1}}, \quad q_k = \frac{d_k - a_k q_{k-1}}{b_k - a_k p_{k-1}}, \quad k = 2,3,\ldots,n \quad (6)$$

Demak, (1) ko'rinishga ega tenglamalar sistemasini yechishda (6) formulalar yordamida haydash koeffitsiyentlari hisoblanadi, so'ng (4) formulalar orqali ketma-ket $x_n, x_{n-1}, \ldots, x_2, x_1$ lar aniqlanadi. Bu metod ikki qismdan iborat: (6) formula yordamida tashkil etiladigan hisoblash jarayoni haydash usulining *to'g'ri yo'li*, (4) bilan bajariladigan hisoblash jarayoni esa haydash usulining *teskari yo'li* deyiladi.

(1) ko'rinishdagi tenglamalar sistemasi yagona yechimga ega bo'lishi uchun yetarli shart $a_k \neq 0$, $c_k \neq 0$, $k = 2,3,\ldots,n-1$ bo'lib, diagonal elementlar salmoqli bo'lishidan iborat, ya'ni $|b_k| \geq |a_k| + |c_k|$, $k = 1,2,\ldots,n$ tengsizliklarning kamida bittasida qa'tiy tengsizlik o'rinli bo'lishi kerak.

5.5-§. Chiziqli algebraik tenglamalar sistemasini yechishda iteratsion metodlar

Iteratsion metodlarni qurishda vektor va matritsalarning normalari va limitlari tushunchalari qatnashadi. Shu sababli ular haqidagi ma'lumotlarni keltiramiz.

x vektorning normasi deb, quyidagi uch shartni qanoatlantiruvchi haqiqiy $\|x\|$ songa aytiladi.

1) $\|x\| \geq 0$ va $x = 0$ bo'lgandagina $\|x\| = 0$;

2) har qanday α son uchun $\|\alpha x\| = |\alpha| \|x\|$;

3) $\|x+y\| \le \|x\|+\|y\|$ uchburchak tengsizligi.

$x=(x_1,x_2,x_3,\cdots x_n)^1$ vektorning yuqoridagi shartlarni qanoatlantiruvchi va ko'proq ishlatiladigan normalardan uchtasini keltiramiz.

1. Kubik norma: $\|x\|_1 = \max\limits_{1\le i\le n}|x_i|$.

2. Oktaedrik norma: $\|x\|_2 = \sum\limits_{i=1}^{n}|x_i|$.

3. Sferik norma: $\|x\|_3 = \sqrt{\sum\limits_{i=1}^{n}|x_i|^2}$.

Bu normalar uchun birinchi va ikkinchi shartlarning bajarilishi bevosita ko'rinib turibdi. Uchinchi shartni tekshiramiz:

Birinchi norma uchun:

$$\|x+y\|_1 = \max_{1\le i\le n}|x_i+y_i| \le \max_{1\le i\le n}|x_i| + \max_{1\le i\le n}|y_i| = \|x\|_1 + \|y\|_1 .$$

Ikkinchi norma uchun:

$$\|x+y\|_2 = \sum_{i=1}^{n}|x_i+y_i| \le \sum_{i=1}^{n}|x_i| + \sum_{i}^{n}|y_i| = \|x\|_2 + \|y\|_2 .$$

Uchinchi norma uchun Koshi-Bunyakovskiy tengsizligidan

$$\|x+y\|_3 = \sqrt{\sum_{i=1}^{n}|x_i+y_i|^2} \le \sqrt{\sum_{i=1}^{n}|x_i|^2} + \sqrt{\sum_{i=1}^{n}|y_i|^2} = \|x\|_3 + \|y\|_3$$

ga ega bo'lamiz.

A kvadrat matritsaning normasi deb, quyidagi shartlarni qanoatlantiruvchi haqiqiy songa aytiladi:

1) $\|A\| \ge 0$ va $A=0$ bo'lganida $\|A\|=0$;

2) ixtiyoriy α soni uchun $\|\alpha A\| = |\alpha|\|A\|$;

3) $\|A+B\| \le \|A\| + \|B\|$;

4) $\|A\cdot B\| \le \|A\|\cdot\|B\|$, xususan $\|A^p\| \le \|A\|^p$, p – natural son,

Agar har qanday kvadrat matritsa A uchun va o'lchami matritsa tartibiga teng bo'lgan ixtiyoriy x vektor uchun

$$\|Ax\| \le \|A\| \cdot \|x\| \qquad (1)$$

tengsizlik bajarilsa, u holda matritsa normasi vektorning berilgan normasi bilan moslashgan deyiladi.

Har qanday A matritsa uchun Ax vektor normasining uzluksizligiga ko'ra [7]

$$\|A\| = \max_{\|x\|=1} \|Ax\| \qquad (2)$$

tenglikda maksimumga erishiladi, ya'ni shunday $x^{(0)}$ vektor topiladiki,

$$\|x^{(0)}\| = 1 \text{ va } \|Ax^{(0)}\| = \|A\|$$

tenglik bajariladi.

(2) tenglik bilan kiritilgan matritsa normasi vektorning berilgan normasiga bo'ysungan deyiladi.

Quyidagi teorema o'rinli bo'ladi [7].

Teorema. Matritsaning bo'ysungan normasi:

1) norma ta'rifining to'rtta shartini qanoatlantiradi;

2) vektorning berilgan normasiga moslashgan;

3) vektorning berilgan normasiga moslangan boshqa har qanday normasidan katta emas.

Vektorlarning yuqorida kiritilgan normalariga bo'ysungan matritsa normalari mos ravishda quyidagilardan iborat:

$$\|A\|_1 = \max_{1\le i\le n} \sum_{k=1}^{n} |a_{ik}| \text{ – kubik norma,} \qquad (3)$$

$$\|A\|_2 = \max_{1\le k\le n} \sum_{i=1}^{n} |a_{ik}| \text{– oktaedrik norma,} \qquad (4)$$

$$\|A\|_3 = \sqrt{\lambda_1} \text{ – sferik norma,} \qquad (5)$$

bu yerda λ_1 $A'A$ matritsaning eng katta xos soni.

Faraz qilaylik,

$$x^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, \cdots x_n^{(k)}\right)' \quad (k = 1, 2, \cdots)$$

vektorlar ketma-ketligi berilgan bo'lsin. Agar n ta chekli

$$x_i = \lim_{k\to\infty} x_i^{(k)} \quad (i = 1, 2, \cdots, n)$$

limitlar mavjud bo'lsa, u holda $x = (x_1, x_2, \cdots x_n)'$ vektor $\{x^{(k)}\}$ *vektorlar ketma-ketligining limiti* deyiladi va bu ketma-ketlikning o'zi x vektorga yaqinlashadi deyiladi.

Shu kabi

$$A^{(k)} = \left[a_{ij}^{(k)}\right] \quad (i, j = 1, 2, \ldots, n;\ k = 1, 2, \ldots)$$

matritsalar ketma-ketligi berilgan bo'lib, n^2 ta chekli $a_{ij} = \lim_{k\to\infty} a_{ij}^{(k)}$ limitlar mavjud bo'lsa, u holda $A = \left[a_{ij}\right]$ matritsa $\{A^{(k)}\}$ *matritsalar ketma-ketligining limiti* deyiladi.

Bu ta'rifga ko'ra, agar matritsalardan tuzilgan cheksiz qator qismiy yig'indilari ketma-ketligining limiti mavjud bo'lsa, u holda qator yaqinlashuvchi deyiladi. Bu limit qatorning yig'indisi deyiladi.

Vektor normasi tushunchasiga asosan vektorlar ketma-ketligining yaqinlashishini quydagicha ta'riflash mumkin.

Agar

$$\lim_{k\to\infty} \left\| x - x^{(k)} \right\| = 0$$

bo'lsa, u holda $\{x^{(k)}\}$ vektorlar ketma-ketligi x vektorga yaqinlashadi, deyiladi.

Bu ta'rif yaqinlashishning oldingi ta'rifiga ekvivalentdir, buni quyidagi teorema izohlaydi.

Teorema 1. Ushbu

$$\lim_{k\to\infty} x^{(k)} = 0$$

bo'lishi uchun

$$\lim_{k\to\infty} \left\| x - x^{(k)} \right\| = 0$$

bo'lishi zarur va yetarlidir.

Xuddi shuningdek, matritsalar uchun ham $\lim_{k\to\infty} A^{(k)} = A$ bo'lishi uchun $\lim_{k\to\infty}\left\|A^{(k)} - A\right\| = 0$ bo'lishi zarur va yetarlidir.

Endi quyidagi

$$E + A + A^2 + \cdots + A^k + \cdots \qquad (6)$$

matritsali geometrik progressiyaning yaqinlashuvchi bo'lishligi shartlarini keltiramiz.

1-lemma. Ushbu

$$\lim_{k\to\infty} A^k = 0$$

o'rinli bo'lishi uchun A matritsaning barcha xos sonlarining modullari birdan kichik bo'lishi zarur va yetarli.

Bu yaqinlashish belgisi amaliy masalalarda ancha noqulaylikka ega, chunki A matritsaning barcha xos sonlari haqida ma'lumot talab qilinadi. Shuning uchun amaliyotda qulayroq bo'lgan quyidagi yetarli shartni keltiramiz.

2-lemma.

$$\lim_{k\to\infty} A^k = 0$$

o'rinli bo'lishi uchun A matritsaning kamida biror normasi birdan kichik bo'lishi yetarlidir.

Isboti. Yuqorida ta'kidlangandek, $\lim_{k\to\infty} A^k = 0$ o'rinli bo'lishi uchun biror normada $\lim_{k\to\infty}\left\|A^k - 0\right\| = 0$ bajarilishi yetarlidir.

Ammo

$$\left\|A^k - 0\right\| = \left\|A^k\right\| \le \left\|A\right\|^k.$$

Demak, biror normada $\|A\| < 1$ bo'lsa, u holda $\lim_{k\to\infty} A^k = 0$ bo'ladi.

3-lemma. Matritsaning barcha xos sonlarining moduli uning normasidan ortmaydi.

Isboti. Xos son tarifiga ko'ra shunday $x \ne 0$ vektor mavjudki, $Ax = \lambda x$ bo'ladi.

Bundan $\|Ax\| = |\lambda|\|x\|$. Lekin $\|Ax\| \le \|A\| \cdot \|x\|$ bo'lganligi uchun $|\lambda| \le \|A\|$.

Yuqorida berilgan (6) matritsali geometrik progressiyaning yaqinlashishiga doir teoremalarni keltiramiz.

Teorema 2. (6) qatorning yaqinlashishi uchun

$$\lim_{k\to\infty} A^k = 0$$

ning bajarilishi zarur va yetarli. Bu holda $E - A$ matritsaning teskarisi mavjud bo'lib

$$E + A + A^2 + \cdots + A^k + \cdots = (E - A)^{-1}$$

tenglik o'rinli bo'ladi.

Isboti. $\lim_{k\to\infty} A^k = 0$ shartning zaruriyligi ayon, chunki sonli qatorlar uchun shunga o'xshash shart zarur bo'lib, n-tartibli matritsaning yaqinlashishi matritsa elementlaridan mos ravishda tuzilgan n^2 ta sonli qatorlarning yaqinlashishiga teng kuchlidir. Yetarliligini ko'rsatamiz va (6) ning yig'indisi $(E - A)^{-1}$ga tengligini topamiz. Agar $\lim_{k\to\infty} A^k = 0$ bo'lsa, u holda 1-lemma ga asosan A matritsaning barcha xos sonlari λ_i lar modullari bo'yicha birdan kichik. Demak, $E - A$ matritsaning xos sonlari $1 - \lambda_i$ $(i = 1, 2, \cdots n)$ bo'lib, ular noldan farqli. Shuning uchun ham $\det(E - A) \ne 0$. Bundan esa $E - A$ matritsaning maxsusmasligi va $(E - A)^{-1}$ ning mavjudligi kelib chiqadi.

Endi

$$(E + A + A^2 + \cdots + A^k)(E - A) = E - A^{k+1}$$

ayniyatni o'ng tomondan $(E - A)^{-1}$ga ko'paytirsak,

$E + A + A^2 + \cdots + A^k = (E - A)^{-1} - A^{k+1}(E - A)^{-1}$ bo'ladi. Bu tenglikda k ni cheksizga intiltirib, limitga o'tamiz va $\lim_{k\to\infty} A^{(k)} = 0$ ni e'tiborga olib

$$E + A + A^2 + \cdots + A^k + \cdots = (E - A)^{-1}$$

ga ega bo'lamiz. Shu bilan isbot tugadi.

(6) qator yaqinlashishini yetarli sharti 2-lemmaga asosan quyidagicha izohlanadi. Bu shart amaliyotda ancha qulaydir.

Teorema 3. (6) qator yaqinlashishi uchun A matritsaning biror normasi birdan kichik bo'lishi yetarlidir.

Quyidagi teorema (6) qatorning yaqinlashishi tezligini aniqlaydi.

Teorema 4. Agar $\|A\| < 1$ bo'lsa, u holda

$$\left\|(E - A)^{-1} - (E + A + A^2 + \cdots + A^k)\right\| \le \frac{\|A\|^{k+1}}{1 - \|A\|}.$$

Isboti. Agar $\|A\| < 1$ bo'lsa, (6) qator $(E - A)^{-1}$ ga yaqinlashadi, shuning uchun ham

$$(E - A)^{-1} - (E + A + A^2 + \cdots + A^k) = A^{k+1} + A^{k+2} + \cdots$$

va

$$\left\|(E - A)^{-1} - (E + A + A^2 + \cdots + A^k)\right\| =$$

$$= \left\|A^{k+1} + A^{k+2} + \cdots\right\| \le \|A\|^{k+1} + \|A\|^{k+2} + \cdots = \frac{\|A\|^{k+1}}{1 - \|A\|}.$$

Teorema isbot bo'ldi.

5.6-§. Oddiy iteratsiya metodi

Faraz qilaylik,

$$Ax = b \tag{1}$$

chiziqli algebraik tenglamalar sistemasi biror usul bilan

$$x = Bx + c \tag{2}$$

ko'rinishiga keltirilgan bo'lsin. Ixtiyoriy $x^{(0)}$ vektor olib uni boshlang'ich yaqinlashish deylik. Agar keyingi yaqinlashishlar

$$x^{(k)} = Bx^{(k)} + c\ ,\ k = 0, 1, \cdots \tag{3}$$

rekurrent formulalar yordamida aniqlansa, bunday metod oddiy iteratsiya metodi deyiladi. Agar (3) ketma-ketlikning limiti x^* mavjud bo'lsa, bu limit (2) sistemaning (shu bilan (1) sistemaning ham) yechimi bo'ladi.

Haqiqatan ham (3) da limitga o'tsak, $x^* = Bx^* + c$ kelib chiqadi.

Oddiy iteratsiya metodining yaqinlashishini quyidagi teorema ko'rsatadi.

Teorema 5. (3) oddiy iteratsion jarayon ixtiyoriy boshlang'ich $x^{(0)}$ da yaqinlashuvchi bo'lishi uchun B matritsaning barcha xos sonlarining modullari birdan kichik bo'lishi zarur va yetarli.

Isboti. Zarurligi. Faraz qilaylik, $\lim\limits_{k\to\infty} x^{(k)} = x^*$ mavjud bo'lsin. U holda $x^* = Bx^* + c$. Bundan (3)ni ayirsak, quyidagilarga ega bo'lamiz:

$$x^* - x^{(k)} = B(x^* - x^{(k-1)}) = B^2(x^* - x^{(k-2)}) = \cdots = B^k(x^* - x^{(0)})$$

Endi $x^* - x^{(0)}$ vektor k ga bog'liq bo'lmaganligi uchun

$$x^* - x^{(k)} = B^k(x^* - x^{(0)})$$

tenglikda $k \to \infty$ limitga o'tsak,

$$\lim_{k\to\infty} B^k = 0$$

kelib chiqadi, bundan 1-lemmaga asosan B matritsaning barcha xos sonlarining modullari birdan kichikligi kelib chiqadi.

Yetarliligi. (3) orqali aniqlangan barcha yaqinlashishlarni boshlang'ich yaqinlashish $x^{(0)}$ va c vektorlar orqali ifodalaymiz:

$$x^{(k)} = Bx^{(k-1)} + c = B(Bx^{(k-2)} + c) + c =$$

$$= B^2x^{(k-2)} + (E + B)c = \cdots = B^kx^{(0)} + (E + B + B^2 + \cdots B^{k-1})c$$

Endi, faraz qilaylik B ning xos sonlarining moduli birdan kichik bo'lsin. U holda 1-lemmaga ko'ra, $\lim\limits_{k\to\infty} B^k = 0$, 2-teoremaga asosan $\lim\limits_{k\to\infty}(E + B + B^2 + \cdots + B^{k-1}) = (E - B)^{-1}$ tengliklar o'rinli bo'ladi. Demak, $x^{(0)}$ qanday bo'lishidan qat'iy nazar $x^{(k)}$ yaqinlashuvchi ketma-ketlik ekan.

Bu teorema nazariy jihatdan foydali, lekin amaliyot uchun yaramaydi. Quyidagi teorema B matritsaning elementlari orqali iteratsion jarayon yaqinlashishining yetarli shartini ko'rsatadi.

Teorema 6. (3) iteratsion jarayon yaqinlashuvchi bo'lishligi uchun B matritsaning biron normasi birdan kichik bo'lishi yetarli.

Isboti. Agar $\|B\| < 1$ bo'lsa, 3-lemmaga asosan B matritsaning barcha xos sonlarining moduli birdan oshmaydi. Bundan 1-teoremaga ko'ra oddiy iteratsion jarayonning yaqinlashuvchiligi kelib chiqadi.

Natija. (3) iteratsion jarayon yaqinlashuvchi bo'lishi uchun B matritsa elementlari quyidagi

$$\max_{1\le i\le n}\sum_{j=1}^{n}|b_{ij}| < 1 \tag{4}$$

$$\max_{1\le j\le n}\sum_{i=1}^{n}|b_{ij}| < 1 \tag{5}$$

$$\sqrt{\sum_{i,j=1}^{n}|b_{ij}|^{2}} < 1 \tag{6}$$

tengsizliklarning birortasini qanoatlantirishi yetarlidir.

(1) tenglamalar sistemasini (2) ko'rinishga keltirish masalasiga to'xtalib o'tamiz.

Agar $a_{ii} \neq 0$ bo'lib, quyidagi

$$\max_{i}\sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ij}| < |a_{ii}| \tag{7}$$

$$\max_{j}\sum_{\substack{i=1\\ i\neq j}}^{n}|a_{ij}| < |a_{ii}| \tag{8}$$

tengsizliklarning birortasi bajarilsa, u holda (2) quyidagi ko'rinishda bo'ladi:

$$\begin{cases} x_1 = 0\cdot x_1 - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \dots - \frac{a_{1n}}{a_{11}}x_n + \frac{b_1}{a_{11}}, \\ x_2 = \frac{a_{21}}{a_{22}}x_1 - 0\cdot x_2 - \frac{a_{23}}{a_{22}}x_3 - \dots - \frac{a_{2n}}{a_{22}}x_n + \frac{b_2}{a_{22}}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n = \frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}}{a_{nn}}x_3 - \frac{a_{n3}}{a_{nn}}x_3 - \dots - 0\cdot x_n + \frac{b_n}{a_{nn}} \end{cases}$$

ya'ni

$$B = \begin{bmatrix} 0 - \frac{a_{12}}{a_{11}} - \frac{a_{13}}{a_{11}} & \dots & -\frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} - 0 - \frac{a_{23}}{a_{22}} & \dots & -\frac{a_{2n}}{a_{22}} \\ \dots & \dots & \dots \\ \frac{a_{n1}}{a_{nn}} - \frac{a_{n2}}{a_{nn}} - \frac{a_{n3}}{a_{nn}} & \dots & 0 \end{bmatrix}$$

$$c = \left(\frac{b_1}{a_{11}}, \frac{b_2}{a_{22}}, \dots, \frac{b_n}{a_{nn}}\right)$$

bo'lib, ixtiyoriy $x^{(0)}$da (3) iteratsion jarayon yaqinlashuvchi bo'ladi.

Agar (7), (8) tengsizliklarning hech qaysisi bajarilmasa, (1) tenglamalar sistemasi ustida shunday chiziqli almashtirishlar qilish kerakki, yuqoridagi tengsizliklarning birortasi bajarilsin. Xususan, berilgan tenglamalardan shundaylari ajratib olinadiki, bu tenglamalarda biror noma'lum oldidagi koeffitsiyent moduli bo'yicha shu tenglamaning qolgan barcha koeffitsiyentlari modullarining yig'indisidan katta bo'lsin. Ajratilgan tenglamalar shunday joylashtiriladiki, ularning moduli bo'yicha eng kattasi diagonal koeffitsiyentlari bo'ladi. Qolgan tenglamalardan va ajratilganlardan yuqoridagi prinsipni saqlagan holda o'zaro chiziqli erkli bo'lgan chiziqli kombinatsiyalar tuziladi va bo'sh satrlar to'ldiriladi. Shu bilan birga boshlang'ich sistemaning har bir tenglamasi yangi sistema tenglamalarini tuzayotganda ishtirok etishi shart.

5.7-§. Zeydel usuli

Faraz qilaylik, bizga

$$x = Bx + c$$

ko'rinishdagi tenglamalar sistemasi berilgan bo'lsin, bu yerda $\|B\| < 1$. Zeydel metodini oddiy iteratsiya metodining qandaydir modifikatsiyasi deb qarash mumkin. Uning asosiy g'oyasi quydagidan iborat: $x_i^{(k+1)}$ noma'lumni topishda $x_1^{(k+1)}, x_2^{(k+1)}, \cdots x_{i-1}^{(k+1)}$ larni ishlatish. Uni quyidagicha yozish mumkin:

$$x_1^{(k+1)} = \sum_{j=1}^{n} b_{1j} x_j^{(k)} + c_1,$$

$$x_2^{(k+1)} = b_{12} x_1^{(k+1)} + \sum_{j=2}^{n} b_{2j} x_j^{(k)} + c_2,$$

....

$$x_i^{(k+1)} = \sum_{j=1}^{i-1} b_{ij} x_j^{(k+1)} + \sum_{j=i}^{n} b_{ij} x_j^{(k)} + c_i$$

....

$$x_n^{(k+1)} = \sum_{j=1}^{n-1} b_{nj} x_j^{(k+1)} + b_{nn} x_n^{(k)} + c_n \quad (k = 0, 1, ...)$$

Oddiy iteratsiya metodining yaqinlashuvchi bo'lishiga doir teorema Zeydel metodida ham o'rinliligini ta'kidlab o'tamiz.

Iteratsion jarayon xatoligini baholash masalasini ko'raylik.

$x = Bx + c$ chiziqli sistema uchun

$$x^{(k+1)} = Bx^{(k)} + c \quad (k = 0, 1, \cdots)$$

iteratsion jarayon bo'lsin. Ikkita ketma-ket yaqinlashishlar orasidagi farqni ko'raylik:

$$x^{(m+1)} - x^{(m)} = Bx^{(m)} + c - (Bx^{(m-1)} + c) = B(x^{(m)} - x^{(m-1)}) = \cdots = B^m(x^{(1)} - x^{(0)}).$$

Buning normasi

$$\|x^{(m+1)} - x^{(m)}\| = \|B^m(x^{(1)} - x^{(0)})\| \le \|B\|^m \|x^{(1)} - x^{(0)}\| \qquad (9)$$

bo'ladi. Endi ixtiyoriy natural $p \geq 1$ uchun $x^{(k+p)} - x^{(k)}$ farqning normasini hisoblaylik.

$$\|x^{(k+p)} - x^{(k)}\| \leq \|x^{(k+1)} - x^{(k)}\| + \|x^{(k+2)} - x^{(k+1)}\| + \cdots + \|x^{(k+p)} - x^{(k+p+1)}\| \leq$$

$$\leq \|B\|^k \|x^{(1)} - x^{(0)}\|\left(1 + \|B\| + \|B\|^2 + \cdots + \|B\|^{p-1}\right) = \frac{1-\|B\|^p}{1-\|B\|} \cdot \|B\|^k \|x^{(1)} - x^{(0)}\|$$

Bu tengsizlikda $p \to \infty$ da limitga o'tsak

$$\|x - x^{(k)}\| \leq \frac{\|B\|^k}{1 - \|B\|}\|x^{(1)} - x^{(0)}\|$$

ga ega bo'lamiz. Bu iteratsion jarayonning yaqinlashish tezligini ifodalaydi.

Chiziqli algebraik tenglamalarni yechishda boshqa usullar haqidagi ma'lumotlarni [2], [5], [7], [13] lardan topish mumkin.

Bobga tegishli tayanch so'zlar: aniq metodlar, iteratsion metodlar, haydash usuli, vektor va matritsalar ketma-ketligi, oddiy iteratsiya, iteratsion jarayon tezligi.

Savollar va topshiriqlar

1. Chiziqli algebraik tenglamalar sistemasini yechishda aniq metodlarni izohlang.
2. Chiziqli algebraik tenglamalar sistemasini yechishda iteratsion metodlarni izohlang.
3. Gauss metodi.
4. Gauss metodida to'g'ri yo'lning barcha qadamlari bajarilmasa bu nimani anglatadi?
5. Kvadrat ildizlar usuli.
6. Matritsasi uch diagonalli chiziqli algebraik tenglamalar sistemasini yechishda haydash usuli.
7. Qanday shartlar o'rinli bo'lganda haydash usulini qo'llash mumkin?
8. Vektor normasining ta'rifi.

9. Vektorning kubik, oktaedrik va sferik normalari.

10. Matritsa normasining ta'rifi.

11. Matritsaning kubik oktaedrik va sferik normalari.

12. Matritsani vektorning berilgan normasi bilan moslashgan normasi.

13. Matritsani vektorning berilgan normasiga bo'ysungan normasi.

14. Oddiy iteratsiya.

15. Oddiy iteratsion jarayon yaqinlashuvchi bo'lishligining zaruriy va yetarli shartlari nimalardan iborat?

16. Oddiy iteratsion jarayon yaqinlashuvchi bo'lishining yetarli sharti nimadan iborat?

17. Zeydel usuli.

Misol 1. Quyidagi chiziqli tenglamalar sistemasini Gauss usuli bilan yeching.

$$\begin{cases} 2x_1 - x_2 + 4x_3 + 3x_4 = 10, \\ 4x_1 + x_2 + 3x_3 - 6x_4 = 0, \\ 6x_1 - 2x_2 + 2x_3 - 4x_4 = 6, \\ 10x_1 + x_2 - 5x_3 + x_4 = 5 \end{cases}$$

Yechish. Berilgan tenglamalar sistemasining kengaytirilgan matritsasini yozamiz va 2–4-tenglamalardan 1-tenglama (yetakchi) yordamida x_1 ni yo'qotish natijasida hosil bo'ladigan kengaytirilgan matritsani aniqlaymiz. Yangi sistemada 2-tenglama (yetakchi) yordamida 3–4-tenglamalardan x_2 yo'qotish natijasida chiqadigan kengaytirilgan matritsani aniqlaymiz va nihoyat hosil bo'lgan sistemaning uchinchi (yetakchi) tenglamasi yordamida to'rtinchidan x_3 ni yo'qotamiz, natijada berilgan sistemaning matritsasi yuqori o'ng uchburchak matritsaga keladi. Undan ketma-ket x_4, x_3, x_2, x_1 larning qiymatini aniqlaymiz. Buni quydagi sxemadagi algoritmda ko'rsatamiz:

$$\left(\begin{array}{cccc|c} 2 & -1 & 4 & 3 & 10 \\ 4 & 1 & 3 & -6 & 0 \\ 6 & -2 & 2 & -4 & 6 \\ 10 & 1 & -5 & 1 & 5 \end{array}\right) \begin{array}{l} I \text{ (yetakchi)} \\ II + I\times(-2) \Rightarrow II \\ III + I(-3) \Rightarrow III \\ IV + I(-5) \Rightarrow IV \end{array} \left(\begin{array}{cccc|c} 2 & -1 & 4 & 3 & 10 \\ 0 & 3 & -5 & -12 & -20 \\ 0 & 1 & -10 & -13 & -24 \\ 0 & 6 & -25 & -14 & -45 \end{array}\right) \Rightarrow$$

$$\begin{array}{l} III + II(-1/3) \Rightarrow III \\ IV + II(-2) \Rightarrow IV \end{array} \left(\begin{array}{cccc|c} 2 & -1 & 4 & 3 & 10 \\ 0 & 3 & -5 & -12 & -20 \\ 0 & 0 & -\frac{25}{3} & -9 & -\frac{52}{3} \\ 0 & 0 & -15 & 10 & -5 \end{array}\right)$$

$$IV + III(-9/5) \Rightarrow IV \Rightarrow \left(\begin{array}{cccc|c} 2 & -1 & 4 & 3 & 10 \\ 0 & 3 & -5 & -12 & -20 \\ 0 & 0 & -\frac{25}{3} & -9 & -\frac{52}{3} \\ 0 & 0 & 0 & \frac{131}{5} & \frac{131}{5} \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{cccc|c} 1 & -\frac{1}{2} & 2 & 1{,}5 & 10 \\ 0 & 1 & -\frac{5}{3} & -4 & -\frac{20}{3} \\ 0 & 0 & 1 & \frac{27}{25} & \frac{52}{25} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{cases} x_1 = 1, \\ x_2 = -1, \\ x_3 = 1, \\ x_4 = 1. \end{cases}$$

Misol 2.
$$\begin{cases} 4x_1 + 4x_2 - 2x_3 = 7, \\ 4x_1 + x_2 + 6x_3 = 8, \\ -2x_1 + 6x_2 + 2x_3 = 5 \end{cases}$$

tenglamalar sistemasini kvadrat ildizlar metodi bilan yeching.

Yechish. Tenglamalar sistemasining matritsasini

$$A = T'T$$

ko'rinishda yozib olamiz, bu yerda T yuqori o'ng uchburchak matritsa, T' esa uning transponirlangani bo'lib, T ning elementlari quyidagicha aniqlangan:

$$t_{11}=\sqrt{a_{11}}=2,\quad t_{12}=\frac{a_{12}}{t_{11}}=2,\quad t_{13}=\frac{a_{13}}{t_{11}}=-\frac{2}{2}=-1,$$

$$t_{22}=\sqrt{a_{22}-t}=\sqrt{1-4}=i\sqrt{3},$$

$$t_{23}=\frac{a_{23}-t_{12}\cdot t_{13}}{t_{22}}=\frac{6-2\cdot(-1)}{i\sqrt{3}}=-\frac{8i}{\sqrt{3}},$$

$$t_{33}=\sqrt{a_{33}-t_{13}^2-t_{23}^2}=\sqrt{2-(-1)^2-\left(\frac{8i}{\sqrt{3}}\right)^2}=\sqrt{\frac{67}{3}}.$$

Endi

$$T'y=\begin{pmatrix}7\\8\\5\end{pmatrix}\Rightarrow\begin{pmatrix}2 & 0 & 0\\ 2 & i\sqrt{3} & 0\\ -1 & -\frac{8i}{\sqrt{3}} & \sqrt{\frac{67}{3}}\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}7\\8\\5\end{pmatrix}$$

tenglamalar sistemasini yechamiz:

$$2y_1=7 \qquad\qquad \Rightarrow y_1=3{,}5,$$

$$2y_1+i\sqrt{3}\,y_2=8 \qquad\qquad \Rightarrow y_2=-\frac{i}{\sqrt{3}},$$

$$-y_1-\frac{8i}{\sqrt{3}}y_2+\sqrt{\frac{67}{3}}y_3=5 \;\Rightarrow\; \sqrt{\frac{67}{3}}y_3=5+3{,}5+\frac{8i}{\sqrt{3}}\cdot\left(-\frac{i}{\sqrt{3}}\right)=\frac{33{,}5}{3},$$

$$y_3=\sqrt{\frac{67}{12}}.$$

Bulardan foydalanib

$$Tx=y\Rightarrow\begin{pmatrix}2 & 2 & -1\\ 0 & i\sqrt{3} & -\frac{8i}{\sqrt{3}}\\ 0 & 0 & \sqrt{\frac{67}{3}}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}3{,}5\\ -\frac{i}{\sqrt{3}}\\ \sqrt{\frac{67}{12}}\end{pmatrix}$$

tenglamalar sistemasidan

$$x_3=\frac{1}{2},\quad x_2\cdot i\sqrt{3}-\frac{8i}{\sqrt{3}}\cdot x_3=-\frac{i}{\sqrt{3}}\Rightarrow x_2=-\frac{1}{3}+\frac{8}{3}\cdot\frac{1}{2}=1,$$

$$2x_1+2x_2-x_3=3{,}5\Rightarrow x_1=\frac{1}{2}\cdot\left(3{,}5\ -2+\frac{1}{2}\right)=1$$

ekanligi kelib chiqadi. $x=(1,1,0,5)'$ berilgan sistemaning aniq yechimi.

Shuni ta'kidlash lozimki, A matritsa simmetrik va musbat aniqlangan bo'lsagina T matritsa diagonal elementlari haqiqiy va musbat bo'ladi.

Misol 3. Quyidagi tenglamalar sistemasini haydash usuli bilan yeching:

$$\begin{cases} 5x_1-3x_2=8, \\ 3x_1+5x_2+x_3=0, \\ x_2+4x_3-2x_4=3, \\ x_3-3x_4=-2. \end{cases}$$

Yechish. Sistemaning kengaytirilgan matritsasini yozamiz:

$$\left(\begin{array}{cccc|c} 5 & 3 & 0 & 0 & 8 \\ 3 & 6 & 1 & 0 & 10 \\ 0 & 1 & 4 & -2 & 3 \\ 0 & 0 & 1 & -3 & -2 \end{array}\right)$$

Diagonal elementlar salmoqliligi ko'rinib turibdi. (6) formula yordamida aniqlanadigan hisoblash jarayoni (to'g'ri yo'l) natijasi quyidagidan iborat:

$$p_1=\frac{c_1}{b_1}=\frac{3}{5},\ p_2=\frac{c_2}{b_2-a_2p_1}=\frac{1}{6-\frac{3\cdot 3}{5}}=\frac{5}{21},$$

$$p_3=\frac{c_3}{b_3-a_3p_2}=\frac{-2}{4-\frac{5}{21}}=\frac{42}{79},\ p_4=\frac{c_4}{b_4-a_4p_3}=0,$$

$$q_1=\frac{d_1}{b_1}=\frac{8}{5},$$

$$q_2=\frac{d_2-a_2q_1}{b_2-a_2p_1}=\frac{10-\frac{3\cdot 8}{5}}{6-\frac{3\cdot 3}{5}}=\frac{26}{21},$$

$$q_3=\frac{d_3-a_3q_2}{b_3-a_3p_2}=\frac{3-\frac{26}{21}}{4-\frac{5}{21}}=\frac{37}{79},$$

$$q_4=\frac{d_4-a_4q_3}{b_4-a_4p_3}=\frac{-2-\frac{37}{79}}{-3-\left(-\frac{42}{79}\right)}=1.$$

(4) yordamida aniqlanadigan hisoblash jarayoni (teskari yo'l) natijalari esa quydagicha:

$$x_4=q_4=1\,,\ x_3=q_3-p_3\cdot x_4=\frac{37}{79}-\left(-\frac{42}{79}\right)1=1,$$

$$x_2=q_2-p_2\cdot x_3=\frac{26}{21}-\frac{5}{21}\cdot 1=1,\ \ x_1=q_1-p_1\cdot x_2=\frac{8}{5}-\frac{3}{5}\cdot 1=1.$$

Berilgan sistemaning aniq yechimi $x=(1,1,1,1)'$.

Misol 4. $Ax=b$ tenglamalar sistemasini iteratsiya metodini qo'llash uchun kanonik ko'rinishga keltiring:

$$A=\begin{bmatrix}10 & 1 & -3 & -2 & 1\\ -1 & 25 & 1 & -5 & -2\\ 2 & 1 & -20 & 2 & -3\\ 0 & 1 & -1 & 10 & -5\\ 1 & 2 & 1 & 2 & -20\end{bmatrix},\ b=\begin{pmatrix}6\\ 11\\ -19\\ 10\\ -32\end{pmatrix}.$$

Yechish. Bu tenglamalar sistemasi matritsasining diagonal elementlari salmoqli bo'lgani uchun uning tenglamalarini mos ravishda $10,\ 25,\ -20, 10,\ -20$ sonlariga bo'lib, uni quyidagi

$$x=Bx+c,$$

ko'rinishga keltiramiz, bu yerda

$$B=\begin{bmatrix} 0 & -0,2 & 0,3 & 0,2 & -0,1 \\ 0,04 & 0 & -0,04 & 0,2 & 0,08 \\ 0,1 & 0,05 & 0 & 0,1 & -0,15 \\ 0 & -0,1 & 0,1 & 0 & 0,5 \\ 0,05 & 0,1 & 0,05 & 0,1 & 0 \end{bmatrix}, \quad c=\begin{pmatrix} 0,6 \\ 0,44 \\ 0,95 \\ 1 \\ 1,6 \end{pmatrix}$$

Bu misol uchun 5.6-§ dagi (4) formula bilan aniqlangan yig'indilar mos ravishda 0,8; 0,36; 0,4; 0,7; 0,3 bo'lib, $\|B\|_1=0,8<1$ ekanligi kelib chiqadi. Demak, ixtiyoriy dastlabki yaqinlashish $x^{(0)}$ olib

$$x^{(k+1)}=Bx^{(k)}+c\,,\quad k=0,1,...$$

jarayonni tuzib, sistemaning taqribiy yechimini berilgan aniqlikda topish mumkin.

Misol 5. Quyidagi sistemani oddiy iteratsiya metodini qo'llash uchun kanonik ko'rinishga keltiring:

$$\begin{aligned} &2x_1-3x_2+6x_3+20x_4=10, && I \\ &x_1+2x_2-15x_3+3x_4=17, && II \\ &-8x_1-x_2+10x_3+19x_4=10, && III \\ &11x_1-9x_2-2x_3-x_4=6, && IV \end{aligned}$$

Yechish. Bu sistemada quyidagi almashtirishni bajaramiz:

$$\begin{aligned} &I-III \Rightarrow && 10x_1-2x_2-4x_3+x_4=0, \\ &IV-I+III \Rightarrow && x_1-7x_2+2x_3-2x_4=6, \\ &II && x_1+2x_2-15x_3+3x_4=7, \\ &I && 2x_1-3x_2+6x_3+20x_4=17. \end{aligned}$$

Hosil bo'lgan sistema matritsasi salmoqli diagonal elementlarga ega, shuning uchun bu sistema tenglamalarini mos ravishda $10, -7, -15, 20$ sonlariga bo'lib, bu sistemani $x=Bx+c$ ko'rinishga keltiramiz, bu yerda $\|B\|_I=0,7<1$ bo'ladi.

Misol 6.

$$\begin{cases} 5x_1 - x_2 + 2x_3 = 8, \\ x_1 + 4x_2 - x_3 = -4, \\ x_1 + x_2 + 4x_3 = 4 \end{cases}$$

tenglamalar sistemasini iteratsiya metodi bilan 10^{-3} aniqlikda yechish uchun nechta iteratsiya o'tkazish kerakligini aniqlang.

Yechish. Berilgan tenglamalar sistemasini quyidagi ko'rinishda yozib olamiz:

$$\begin{cases} x_1 = 0\cdot x_1 + 0,2\cdot x_2 - 0,4\cdot x_3 + 1,6, \\ x_2 = -0,25\cdot x_1 + 0\cdot x_2 + 0,25\cdot x_3 - 1, \\ x_3 = -0,25\cdot x_1 - 0,25\cdot x_2 + 0\cdot x_3 + 1. \end{cases} \quad (1)$$

Quyidagi iteratsion jarayonni yozamiz:

$$\begin{cases} x_1^{(k+1)} = 0\cdot x_1^{(k)} + 0,2\cdot x_2^{(k)} - 0,4\cdot x_3^{(k)} + 1,6, \\ x_2^{(k+1)} = -0,25\cdot x_1^{(k)} + 0\cdot x_2^{(k)} + 0,25\cdot x_3^{(k)} - 1, \\ x_3^{(k+1)} = -0,25\cdot x_1^{(k)} - 0,25\cdot x_2^{(k)} + 0\cdot x_3^{(k)} + 1. \end{cases} \quad (2)$$

Agar boshlang'ich yaqinlashishni $x^{(0)} = \beta$ (ozod hadlar ustuni) desak, xatolik bahosi quyidagi

$$\left\| x - x^{(k)} \right\| \le \frac{\|\alpha\|^{k+1}}{1 - \|\alpha\|} \max_{1 \le i \le n} |\beta_i|$$

ko'rinishda bo'ladi. (1) sistemaning matritsasi

$$\alpha = \begin{pmatrix} 0 & 0,2 & -0,4 \\ -0,25 & 0 & 0,25 \\ -0,25 & -0,25 & 0 \end{pmatrix}$$

ko'rinishga ega bo'lib, uning normasi

$$\|\alpha\| \le \max_{1 \le i \le 3} \left[\sum_{j=1}^{3} |a_{ij}| \right] = \max[0,6; 0,5; 0,5] = 0,6$$

ga teng. $\max\limits_{1 \le i \le 3} |\beta_i| = \max\limits_{1 \le i \le 3} \{ 1,6; |-1|; 1 \} = 1,6.$

Berilgan aniqlikka erishish uchun o‘tkaziladigan iteratsiyalar sonini esa

$$\frac{\|\alpha\|^{k+1}}{1-\|\alpha\|}\max_{1\le i\le 3}|\beta_i|\le\varepsilon=10^{-3}$$

tengsizlikdan topiladi, uni

$$(k+1)\lg\|\alpha\|\le\lg\left(\frac{10^{-3}\cdot 1-\|\alpha\|}{\max\limits_{1\le i\le 3}|\beta_i|}\right)$$

ko‘rinishda yozib olamiz. $\|\alpha\|=0,6$, $\max\limits_{1\le i\le n}|\beta_i|=1,6$ ekanligini e’tiborga olsak,

$$k\ge\frac{-3+\lg 0,25}{\lg 0,6}-1\approx 15$$

bo‘ladi. Quyida ketma-ket iteratsiya natijalarini keltiramiz.

Boshlang‘ich yaqinlashish – $x^{(0)}=(1,6;-1,1)$,

1-iteratsiya – $x^{(1)}=(1;-1,5;0,85)'$;

2-iteratsiya – $x^{(2)}=(1,11;-1,0375;1,0375)'$;

3-iteratsiya – $x^{(3)}=(0,9775;-0,9794;0,9794)'$.

4-iteratsiya – $x^{(4)}=(0,9925;-0,9995;0,9995)'$.

Misol 7. Oldingi 6-misolda $x^{(0)}=(1,6;-1,1)'$ deb, Zeydel metodi bo‘yicha sistemani yeching:

$$x_1^{(1)}=0\cdot x_1^{(0)}+0,2x_2^{(0)}-0,4x_3^{(0)}+1,6=0,2(-1)-0,4\cdot 1+1,6=1,$$
$$x_2^{(1)}=-0,25\cdot x_1^{(1)}+0\cdot 2x_2^{(0)}+0,25x_3^{(0)}-1=0,25\cdot 1+0,25\cdot 1-1=-1,$$
$$x_3^{(1)}=-0,25\cdot x_1^{(1)}+0,2\cdot x_2^{(1)}+0\cdot x_3^{(0)}+1=-0,25\cdot 1-0,25\cdot(-1)+1=1.$$

$\left(x_1^{(1)},x_2^{(1)},x_3^{(1)}\right)'=\left(1,-1,1\right)'$ berilgan sistemaning aniq yechimi. Zeydel metodi bir qadamda aniq yechim qiymatini beradi.

Misollar.

Tenglamalar sistemasini oddiy iteratsiya usuli bilan 0,001 aniqlikda yeching.

№1. $$\begin{cases} x_1 = 0,23x_1 - 0,04x_2 + 0,21x_3 - 0,18x_4 + 1,24; \\ x_2 = 0,45x_1 - 0,23x_2 + 0,06x_3 - 0,88; \\ x_3 = 0,26x_1 + 0,34x_2 - 0,11x_3 + 0,62; \\ x_4 = 0,05x_1 - 0,26x_2 + 0,34x_3 - 0,12x_4 - 1,17. \end{cases}$$

№2. $$\begin{cases} x_1 = 0,21x_1 + 0,12x_2 - 0,34x_3 - 0,16x_4 - 0,64; \\ x_2 = 0,34x_1 - 0,08x_2 + 0,17x_3 - 0,18x_4 + 1,42; \\ x_3 = 0,16x_1 + 0,34x_2 + 0,15x_3 - 0,31x_4 - 0,42; \\ x_4 = 0,12x_1 - 0,26x_2 - 0,08x_3 + 0,25x_4 + 0,83. \end{cases}$$

№3. $$\begin{cases} x_1 = 0,32x_1 - 0,18x_2 + 0,02x_3 + 0,21x_4 + 1,83; \\ x_2 = 0,16x_1 + 0,12x_2 - 0,14x_3 + 0,27x_4 - 0,65; \\ x_3 = 0,37x_1 + 0,27x_2 - 0,02x_3 - 0,24x_4 + 2,23; \\ x_4 = 0,12x_1 + 0,21x_2 - 0,18x_3 + 0,25x_4 - 1,13. \end{cases}$$

№4. $$\begin{cases} x_1 = 0,42x_1 - 0,32x_2 + 0,03x_3 + 0,44; \\ x_2 = 0,11x_1 - 0,26x_2 - 0,36x_3 + 1,42; \\ x_3 = 0,12x_1 + 0,08x_2 - 0,14x_3 - 0,24x_4 - 0,83; \\ x_4 = 0,15x_1 - 0,35x_2 - 0,18x_3 - 1,42. \end{cases}$$

№5. $$\begin{cases} x_1 = 0,18x_1 - 0,34x_2 - 0,12x_3 + 0,15x_4 - 1,33; \\ x_2 = 0,11x_1 + 0,23x_2 - 0,15x_3 + 0,32x_4 + 0,84; \\ x_3 = 0,05x_1 - 0,12x_2 + 0,14x_3 - 0,18x_4 - 1,16; \\ x_4 = 0,12x_1 + 0,08x_2 + 0,06x_3 + 0,57. \end{cases}$$

№6. $$\begin{cases} x_1 = 0,13x_1 + 0,23x_2 - 0,44x_3 - 0,05x_4 + 2,13; \\ x_2 = 0,24x_1 - 0,31x_3 + 0,15x_4 - 0,18; \\ x_3 = 0,06x_1 + 0,15x_2 - 0,23x_4 + 1,44; \\ x_4 = 0,72x_1 - 0,08x_2 - 0,05x_3 + 2,42. \end{cases}$$

$$№7.\ \begin{cases} x_1 = 0{,}17x_1 + 0{,}31x_2 - 0{,}18x_3 + 0{,}22x_4 - 1{,}71; \\ x_2 = -0{,}21x_1 + 0{,}33x_3 + 0{,}22x_4 + 0{,}62; \\ x_3 = 0{,}32x_1 - 0{,}18x_2 + 0{,}05x_3 - 0{,}19x_4 - 0{,}89; \\ x_4 = 0{,}12x_1 + 0{,}28x_2 - 0{,}14x_3 + 0{,}94. \end{cases}$$

$$№8.\ \begin{cases} x_1 = 0{,}13x_1 + 0{,}27x_2 - 0{,}22x_3 - 0{,}18x_4 + 1{,}21; \\ x_2 = -0{,}21x_1 - 0{,}45x_3 + 0{,}18x_4 - 0{,}33; \\ x_3 = 0{,}12x_1 + 0{,}13x_2 - 0{,}33x_3 + 0{,}18x_4 - 0{,}48; \\ x_4 = 0{,}33x_1 - 0{,}05x_2 + 0{,}06x_3 - 0{,}28x_4 - 0{,}17. \end{cases}$$

$$№9.\ \begin{cases} x_1 = 0{,}19x_1 - 0{,}07x_2 + 0{,}38x_3 - 0{,}21x_4 - 0{,}81; \\ x_2 = -0{,}22x_1 + 0{,}08x_2 + 0{,}11x_3 + 0{,}33x_4 - 0{,}64; \\ x_3 = 0{,}51x_1 - 0{,}07x_2 + 0{,}09x_3 - 0{,}11x_4 + 1{,}71; \\ x_4 = 0{,}33x_1 - 0{,}41x_2 - 1{,}21. \end{cases}$$

$$№10.\ \begin{cases} x_1 = 0{,}22x_2 - 0{,}11x_3 + 0{,}31x_4 + 2{,}7; \\ x_2 = 0{,}38x_1 - 0{,}12x_3 + 0{,}22x_4 - 1{,}5; \\ x_3 = 0{,}11x_1 + 0{,}23x_2 - 0{,}51x_4 + 1{,}2; \\ x_4 = 0{,}17x_1 - 0{,}21x_2 + 0{,}31x_3 - 0{,}17. \end{cases}$$

$$№11.\ \begin{cases} x_1 = 0{,}07x_1 - 0{,}08x_2 + 0{,}11x_3 - 0{,}18x_4 - 0{,}51; \\ x_2 = 0{,}18x_1 + 0{,}52x_2 + 0{,}21x_4 + 1{,}17; \\ x_3 = 0{,}13x_1 + 0{,}31x_2 - 0{,}21x_4 - 1{,}02; \\ x_4 = 0{,}08x_1 - 0{,}33x_3 + 0{,}28x_4 - 0{,}28. \end{cases}$$

$$№12.\ \begin{cases} x_1 = 0{,}05x_1 - 0{,}06x_2 - 0{,}12x_3 + 0{,}14x_4 - 2{,}17; \\ x_2 = 0{,}04x_1 - 0{,}12x_2 + 0{,}08x_3 + 0{,}11x_4 + 1{,}4; \\ x_3 = 0{,}34x_1 + 0{,}08x_2 - 0{,}06x_3 + 0{,}14x_4 - 2{,}1; \\ x_4 = 0{,}11x_1 + 0{,}12x_2 - 0{,}03x_4 - 0{,}8. \end{cases}$$

$$№13.\ \begin{cases} x_1 = 0{,}08x_1 - 0{,}03x_2 - 0{,}04x_4 - 1{,}2; \\ x_2 = 0{,}31x_2 + 0{,}27x_3 - 0{,}08x_4 + 0{,}81; \\ x_3 = 0{,}33x_1 - 0{,}07x_3 + 0{,}21x_4 - 0{,}92; \\ x_4 = 0{,}11x_1 + 0{,}03x_3 + 0{,}58x_4 + 0{,}17. \end{cases}$$

№14. $\begin{cases} x_1 = 0{,}12x_1 - 0{,}23x_2 + 0{,}25x_3 - 0{,}16x_4 + 1{,}24; \\ x_2 = 0{,}14x_1 + 0{,}34x_2 - 0{,}18x_3 + 0{,}24x_4 - 0{,}89; \\ x_3 = 0{,}33x_1 + 0{,}03x_2 + 0{,}16x_3 - 0{,}32x_4 + 1{,}15; \\ x_4 = 0{,}12x_1 - 0{,}05x_2 + 0{,}15x_4 - 0{,}57. \end{cases}$

№15. $\begin{cases} x_1 = 0{,}23x_1 - 0{,}14x_2 + 0{,}06x_3 - 0{,}12x_4 + 1{,}21; \\ x_2 = 0{,}12x_1 + 0{,}32x_3 - 0{,}18x_4 - 0{,}72; \\ x_3 = 0{,}08x_1 - 0{,}12x_2 + 0{,}23x_3 + 0{,}32x_4 - 0{,}58; \\ x_4 = 0{,}25x_1 + 0{,}22x_2 + 0{,}14x_3 + 1{,}56. \end{cases}$

№16. $\begin{cases} x_1 = 0{,}14x_1 + 0{,}23x_2 + 0{,}18x_3 + 0{,}17x_4 - 1{,}42; \\ x_2 = 0{,}12x_1 - 0{,}14x_2 + 0{,}08x_3 + 0{,}09x_4 - 0{,}83; \\ x_3 = 0{,}16x_1 + 0{,}24x_2 - 0{,}35x_4 + 1{,}21; \\ x_4 = 0{,}23x_1 - 0{,}08x_2 + 0{,}05x_3 + 0{,}25x_4 + 0{,}65. \end{cases}$

№17. $\begin{cases} x_1 = 0{,}24x_1 + 0{,}21x_2 + 0{,}06x_3 - 0{,}34x_4 + 1{,}42; \\ x_2 = 0{,}05x_1 + 0{,}32x_3 + 0{,}12x_4 - 0{,}57; \\ x_3 = 0{,}35x_1 - 0{,}27x_2 - 0{,}05x_4 + 0{,}68; \\ x_4 = 0{,}12x_1 - 0{,}43x_2 + 0{,}04x_3 - 0{,}21x_4 - 2{,}14. \end{cases}$

№18. $\begin{cases} x_1 = 0{,}17x_1 + 0{,}27x_2 - 0{,}13x_3 - 0{,}11x_4 - 1{,}42; \\ x_2 = 0{,}13x_1 - 0{,}12x_2 + 0{,}09x_3 - 0{,}06x_4 + 0{,}48; \\ x_3 = 0{,}11x_1 + 0{,}05x_2 - 0{,}02x_3 + 0{,}12x_4 - 2{,}34; \\ x_4 = 0{,}13x_1 + 0{,}18x_2 + 0{,}24x_3 + 0{,}43x_4 + 0{,}72. \end{cases}$

№19. $\begin{cases} x_1 = 0{,}15x_1 + 0{,}05x_2 - 0{,}08x_3 + 0{,}14x_4 - 0{,}48; \\ x_2 = 0{,}32x_1 - 0{,}13x_2 - 0{,}12x_3 + 0{,}11x_4 + 1{,}24; \\ x_3 = 0{,}17x_1 + 0{,}06x_2 - 0{,}08x_3 + 0{,}12x_4 + 1{,}15; \\ x_4 = 0{,}21x_1 - 0{,}16x_2 + 0{,}36x_3 - 0{,}88. \end{cases}$

№20. $\begin{cases} x_1 = 0{,}28x_2 - 0{,}17x_3 + 0{,}06x_4 + 0{,}21; \\ x_2 = 0{,}52x_1 + 0{,}12x_3 + 0{,}17x_4 - 1{,}17; \\ x_3 = 0{,}17x_1 - 0{,}18x_2 + 0{,}21x_3 - 0{,}81; \\ x_4 = 0{,}11x_1 + 0{,}22x_2 + 0{,}03x_3 + 0{,}05x_4 + 0{,}72. \end{cases}$

№21. $\begin{cases} x_1 = 0{,}52x_2 + 0{,}08x_3 + 0{,}13x_4 - 0{,}22; \\ x_2 = 0{,}07x_1 - 0{,}38x_2 - 0{,}05x_3 + 0{,}41x_4 + 1{,}8; \\ x_3 = 0{,}04x_1 + 0{,}42x_2 + 0{,}11x_3 - 0{,}07x_4 - 1{,}3; \\ x_4 = 0{,}17x_1 + 0{,}18x_2 - 0{,}13x_3 + 0{,}19x_4 + 0{,}33. \end{cases}$

№22. $\begin{cases} x_1 = 0{,}01x_1 + 0{,}02x_2 - 0{,}62x_3 + 0{,}08x_4 - 1{,}3; \\ x_2 = 0{,}03x_1 + 0{,}28x_2 + 0{,}33x_3 - 0{,}07x_4 + 1{,}1; \\ x_3 = 0{,}09x_1 + 0{,}13x_2 + 0{,}42x_3 + 0{,}28x_4 - 1{,}7; \\ x_4 = 0{,}19x_1 - 0{,}23x_2 + 0{,}08x_3 + 0{,}37x_4 + 1{,}5. \end{cases}$

№23. $\begin{cases} x_1 = 0{,}17x_2 - 0{,}33x_3 + 0{,}18x_4 - 1{,}2; \\ x_2 = 0{,}18x_2 + 0{,}43x_3 - 0{,}08x_4 + 0{,}33; \\ x_3 = 0{,}22x_1 + 0{,}18x_2 + 0{,}21x_3 + 0{,}07x_4 + 0{,}48; \\ x_4 = 0{,}08x_1 + 0{,}07x_2 + 0{,}21x_3 + 0{,}04x_4 - 1{,}2. \end{cases}$

№24. $\begin{cases} x_1 = 0{,}03x_1 - 0{,}05x_2 + 0{,}22x_3 - 0{,}33x_4 + 0{,}43; \\ x_2 = 0{,}22x_1 + 0{,}55x_2 - 0{,}08x_3 + 0{,}07x_4 - 1{,}8; \\ x_3 = 0{,}33x_1 + 0{,}13x_2 - 0{,}08x_3 - 0{,}05x_4 - 0{,}8; \\ x_4 = 0{,}08x_1 + 0{,}17x_2 + 0{,}29x_3 + 0{,}33x_4 + 1{,}7. \end{cases}$

№25. $\begin{cases} x_1 = 0{,}13x_1 + 0{,}22x_2 - 0{,}33x_3 + 0{,}07x_4 + 0{,}11; \\ x_2 = 0{,}45x_2 - 0{,}23x_3 + 0{,}07x_4 - 0{,}33; \\ x_3 = 0{,}11x_1 - 0{,}08x_3 + 0{,}18x_4 + 0{,}85; \\ x_4 = 0{,}08x_1 + 0{,}09x_2 + 0{,}33x_3 + 0{,}21x_4 - 1{,}7. \end{cases}$

№26. $\begin{cases} x_1 = 0{,}32x_1 - 0{,}16x_2 - 0{,}08x_3 + 0{,}15x_4 + 2{,}42; \\ x_2 = 0{,}16x_1 - 0{,}23x_2 + 0{,}11x_3 - 0{,}21x_4 + 1{,}43; \\ x_3 = 0{,}05x_1 - 0{,}08x_2 + 0{,}34x_4 - 0{,}16; \\ x_4 = 0{,}12x_1 + 0{,}14x_2 - 0{,}18x_3 + 0{,}06x_4 + 1{,}62. \end{cases}$

№27. $\begin{cases} x_1 = 0{,}08x_2 - 0{,}23x_3 + 0{,}32x_4 + 1{,}34; \\ x_2 = 0{,}16x_1 - 0{,}23x_2 + 0{,}18x_3 + 0{,}16x_4 - 2{,}33; \\ x_3 = 0{,}15x_1 + 0{,}12x_2 + 0{,}32x_3 - 0{,}18x_4 + 0{,}34; \\ x_4 = 0{,}25x_1 + 0{,}21x_2 - 0{,}16x_3 + 0{,}03x_4 + 0{,}63. \end{cases}$

№28. $\begin{cases} x_1 = 0,06x_1 + 0,18x_2 + 0,33x_3 + 0,16x_4 + 2,43; \\ x_2 = 0,32x_1 + 0,23x_3 - 0,05x_4 - 1,12; \\ x_3 = 0,16x_1 - 0,08x_2 - 0,12x_4 + 0,43; \\ x_4 = 0,09x_1 + 0,22x_2 - 0,13x_3 + 0,83. \end{cases}$

№29. $\begin{cases} x_1 = 0,34x_2 + 0,23x_3 - 0,06x_4 + 1,42; \\ x_2 = 0,11x_1 - 0,23x_2 - 0,18x_3 + 0,36x_4 - 0,66; \\ x_3 = 0,23x_1 - 0,12x_2 + 0,16x_3 - 0,35x_4 + 1,08; \\ x_4 = 0,12x_1 + 0,12x_2 - 0,47x_3 + 0,18x_4 + 1,72. \end{cases}$

№30.

VI BOB. MATRITSANING XOS SON VA XOS VEKTORLARINI HISOBLASH

6.1-§. Umumiy mulohazalar

Nazariy va amaliy masalalarni yechishda ko'pincha matritsaning xos sonlarini topish talab qilinadi. Masalan, chiziqli algebraik tenglamalar sitemasini iteratsion metod bilan yechishda va bu metodning yaqinlashishi va yaqinlashish tezligi matritsaning moduli bo'yicha eng katta xos sonining miqdoriga bog'liq edi.

Agar biror x vektor uchun

$$Ax = \lambda x$$

tenglik bajarilsa u holda λ son *A kvadrat matritsaning xos soni* deyiladi. Bu tenglikni qanoatlantiradigan noldan farqli x vektor *A matritsaning λ soniga mos keladigan xos vektori* deyiladi.

$$D(\lambda)=\det(A-\lambda E)=\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n-1} & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn-1} & a_{nn}-\lambda \end{vmatrix}=0 \quad (1)$$

tenglama *A matritsaning xarakteristik tenglamasi* deyiladi.

$$\det(A-\lambda E)=(-1)^n\left(\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - p_3\lambda^{n-3} - \dots - p_n\right) \quad (2)$$

A matritsaning xos yoki xarakteristik ko'phadi deyiladi.

Matritsaning barcha xos sonlari va ularga mos xos vektorlarni topish masalasi *xos qiymatlarni to'liq muammosi* deyiladi. Xos sonlarning bittasi yoki ularning bir qismini va mos ravishda xos vektorini topish *xos qiymatlarning qismiy muammosi* deyiladi.

Agar A matritsaning barcha xos sonlarini topish masalasi qo'yilgan bo'lsa, u holda uning xarakteristik tenglamasi $D(\lambda)=0$ ni tuzish kerak bo'ladi. Buning uchun (1)dagi determinantni hisoblash lozim.

Algebradan ma'lumki, (2) xarakteristik ko'phadning koeffitsiyentlari p_i lar A matritsaning $(-1)^{i-1}$ ishora bilan olingan i-tartibli bosh minorlarining yig'indisiga teng:

$$p_1 = \sum_{j=1}^{n} a_{jj}, \quad p_2 = -\sum_{j<k} \begin{vmatrix} a_{jj} & a_{jk} \\ a_{kj} & a_{kk} \end{vmatrix}, \quad p_3 = -\sum_{j<k<l} \begin{vmatrix} a_{jj} & a_{jk} & a_{jl} \\ a_{kj} & a_{kk} & a_{kl} \\ a_{lj} & a_{lk} & a_{ll} \end{vmatrix} \qquad (3)$$

va hokazo. Demak

$$p_n = (-1)^{n-1} \det A.$$

(4)

Bundan ko'rinadiki, A matritsaning i-tartibli bosh minorlarining soni C_n^i ga teng. Matritsaning tartibi n bo'lganligi uchun (2) ko'phadning koeffitsiyentlarini topishda tartiblari har xil bo'lgan

$$\sum_{i=1}^{n} C_n^i = 2^n - 1$$

ta determenantlarni hisoblash kerak. Yetarli katta n uchun bu masala katta hisoblashlarni talab etadi.

Viet teoremasidan foydalanib, quyidagi

$$\begin{aligned} &\lambda_1 + \lambda_2 + \cdots + \lambda_n = p_1 \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ &\lambda_1 \cdot \lambda_2 \cdots \lambda_n = (-1)^{n-1} p_n \end{aligned} \qquad (5)$$

tengliklarni hosil qilamiz. Buni (3)ning birinchisi va (4) bilan taqqoslasak,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn} = Sp\, A,$$
$$\lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det A$$

kelib chiqadi.

Bundan, xususan, quyidagi kelib chiqadi: matritsaning birorta xos soni nolga teng bo'lishi uchun uning determinanti nolga teng bo'lishi zarur va yetarlidir.

6.2-§. Krilov metodi

A matritsaning xarakteristik tenglamasini quyidagicha yozaylik:

$$\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - p_3\lambda^{n-3} - \cdots - p_n = 0. \qquad (1)$$

Ma'lumki (Gamilton-Keli teoremasi), har qanday matritsa o'zining xarakteristik tenglamasini qanoatlantiradi, ya'ni

$$A^n - p_1 A^{n-1} - p_2 A^{n-2} - p_3 A^{n-3} - \cdots - p_n E = 0. \qquad (2)$$

Endi ixtiyoriy noldan farqli $y^{(0)}$ vektor olamiz va $y^{(k)} = A^k y^{(0)}$, $k = 0,1,\ldots,n$ vektorlarni hosil qilamiz. (2) ni o'ngdan $y^{(0)}$ ga ko'paytirsak, quyidagi vektor tenglama hosil bo'ladi:

$$p_1 y^{(n-1)} + p_2 y^{(n-2)} + \cdots + p_n y^{(0)} = y^{(n)}$$

Bu ifodani ochib yozaylik:

$$\begin{cases} p_1 y_1^{(n-1)} + p_2 y_1^{(n-2)} + \cdots + p_n y_1^{(0)} = y_1^{(n)}, \\ p_1 y_2^{(n-1)} + p_2 y_2^{(n-2)} + \cdots + p_n y_2^{(0)} = y_2^{(n)}, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ p_1 y_n^{(n-1)} + p_2 y_n^{(n-2)} + \cdots + p_n y_n^{(0)} = y_n^{(n)}. \end{cases} \qquad (3)$$

(3) tenglamalar sistemasining determinanti faqat $y^{(n-1)}, y^{(n-2)}, \ldots, y^{(0)}$ vektorlar chiziqli erkli bo'lgandagina noldan farqli bo'ladi, chunki uning ustunlari shu vektor kordinatalaridan tuzilgan.

Agar (3)ni yechishda Gauss metodining to'g'ri yurishidagi barcha n ta qadam bajarilgan bo'lsa, (3) sistema quyidagi

$$\begin{cases} p_1 + b_{12}p_2 + b_{13}p_3 + \cdots + b_{1n}p_n = d_1, \\ \qquad p_2 + b_{23}p_3 + \cdots + b_{2n}p_n = d_2, \\ \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \qquad\qquad\qquad p_n = d_n \end{cases} \qquad (4)$$

uchburchak shaklga kelgan bo'ladi va (3) ning determinanti noldan farqli bo'ladi. Demak, (4) dan ketma-ket $p_n, p_{n-1}, \ldots, p_1$ lar aniqlanadi, ya'ni (1) tenglamaning koeffitsiyentlari topilgan bo'ladi. Bu tenglamani yechib, $\lambda_1,\lambda_2,\ldots,\lambda_n$ lar, ya'ni A matritsaning xos sonlari topiladi.

Endi xos vektorlar x_i $(i=1,2,...,n)$ larni topish masalasini ko'ramiz. Buning uchun $y^{(k)}(k=0,1,\ldots,n-1)$ larni $x^{(i)}$ vektorlar orqali yoyib olamiz:

$$y^{(k)}=\sum_{i=1}^{n}c_iA^kx^{(i)}=\sum_{i=1}^{n}c_i\lambda_i^kx^{(i)},\ k=0,1,\ldots,n-1. \qquad (5)$$

Quyidagi ko'phadni tuzamiz:

$$\varphi_i(\lambda)=\lambda^{n-1}+q_{1i}\lambda^{n-2}+q_{2i}\lambda^{n-3}+\cdots+q_{n-1i},\ i=1,2,\ldots,n \qquad (6)$$

$y^{(k)}$, $(k=0,1,2,\ldots,n-1)$ vektorlarni mos ravishda $q_{n-1i}, q_{n-2i}, \ldots, q_{1i}, 1$ larga ko'paytirib chiziqli kombinatsiyasini yozamiz va (5)–(6)larni e'tiborga olsak quyidagiga

$$\begin{gathered}y^{(n-1)}+q_{1i}y^{(n-2)}+q_{2i}y^{(n-3)}+\cdots+q_{n-1i}y^{(0)}=c_1\varphi_i(\lambda_1)x^{(1)}+\\+c_2\varphi_i(\lambda_2)x^{(2)}+\cdots+c_n\varphi_i(\lambda_n)x^{(n)}\end{gathered} \qquad (7)$$

ega bo'lamiz.

Agar $\varphi_i(\lambda)=\dfrac{D(\lambda)}{\lambda-\lambda_i}, i=1,2,3,\ldots,n$ desak, $\varphi_i(\lambda_j)=0,\ i\neq j$ bo'lganligi uchun (7) ifoda

$$y^{(n-1)}+q_{1i}y^{(n-2)}+q_{2i}y^{(n-3)}+\cdots+q_{n-1i}y^{(0)}=c_i\varphi_i(\lambda_i)x^{(i)},\ i=1,2,\ldots,n$$

ko'rinishga keladi. Demak, A matritsaning $x^{(i)}$ xos vektori noldan farqli ko'paytuvchi miqdorida aniqlangan bo'ldi. q_{ji} koeffitsiyentlar esa xarakteristik ko'phadning koeffitsiyentlari orqali

$$\begin{aligned}&q_{0i}=1,\\&q_{ji}=\lambda_iq_{j-1i}+p_j,\ j=1,2,\ldots,n-1.\end{aligned}$$

rekurrent formula yordamida topiladi.

Agar (3) tenglamalar sistemasini yechishda Gauss metodining to'g'ri yo'lini faqat m ta $(m<n)$ qadami bajarilsa, u holda

$y^{(0)}, y^{(1)}, \ldots, y^{(m-1)}$ vektorlar chiziqli erklidir. Shuning uchun (3) tenglamalar o‘rniga quyidagi

$$\begin{cases} p_1 y_1^{(m-1)} + p_2 y_1^{(m-2)} + \cdots + p_m y_1^{(0)} = y_1^{(m)}, \\ p_1 y_2^{(m-1)} + p_2 y_2^{(m-2)} + \cdots + p_m y_2^{(0)} = y_2^{(m)}, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ p_1 y_n^{(m-1)} + p_2 y_n^{(m-2)} + \cdots + p_m y_n^{(0)} = y_n^{(m)} \end{cases}$$

tenglamalar sistemasidan m ta chiziqli erkli tenglamalarni ajratib olib, $p_m, p_{m-1}, \ldots, p_1$ koeffitsiyentlarni topamiz. So‘ng $\lambda^m + p_1\lambda^{m-1} + p_2\lambda^{m-2} + + p_3\lambda^{m-3} + \cdots + p_m = 0$ tenglamadan $\lambda_1, \lambda_2, \ldots, \lambda_m$ larni topamiz. $\lambda^m + p_1\lambda^{m-1} + p_2\lambda^{m-2} + p_3\lambda^{m-3} + \cdots + p_m$ ko‘phad A *matritsaning minimal ko‘phadi* deyiladi.

Xos vektor esa quyidagicha topiladi:

$$x^{(i)} = \beta_{i1} y^{(0)} + \ldots + \beta_{im} y^{(m-1)}, \; i = 1, 2, \ldots, m,$$

bu yerda

$$\beta_{im} = 1$$
$$\beta_{im-1} = \lambda_i - p_1$$
$$\beta_{im-2} = \lambda_i^2 - p_1\lambda_i - p_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\beta_{i1} = \lambda_i^{m-1} - p_1\lambda_i^{m-2} - \ldots - p_{m-1}.$$

6.3-§. Danilevskiy metodi

Bu metodning asosiy g‘oyasi berilgan A matritsani o‘xshash almashtirishlar yordamida Frobenius

$$P = \begin{vmatrix} p_1 & p_2 & p_3 & \cdots & p_{n-1} & p_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{vmatrix}$$

normal formasiga keltirishdan iboratdir. A va R o'xshash bo'lganligi uchun, ya'ni $P = S^{-1}AS$ ular bir xil xarakteristik ko'phadga ega, ya'ni

$$\det(A - \lambda E) = \det(P - \lambda E).$$

R matritsaning xarakteristik ko'phadini osongina yozish mumkin. Haqiqatan ham

$$\det(P - \lambda E) = \begin{vmatrix} p_1 - \lambda & p_2 & p_3 & \dots & p_{n-1} & p_n \\ 1 & -\lambda & 0 & \dots & 0 & 0 \\ 0 & 1 & -\lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & -\lambda \end{vmatrix}$$

ni birinchi satr elementlari bo'yicha yoyib chiqsak:

$$\det(P - \lambda E) = (R_1 - \lambda)(-\lambda)^{n-1} - R_2(-\lambda)^{n-2} + R_3(-\lambda)^{n-3} + \dots + (-1)^{n-1}R_n =$$
$$= (-1)^n(\lambda^n - R_1\lambda^{n-1} - R_2\lambda^{n-2} - \dots - R_n)$$

bo'ladi. Demak, R matritsaning birinchi satr elementlari R_1, R_2, R_3, ..., R_n lar mos ravishda uning xos ko'phadining koeffitsiyentlaridan iborat ekan.

A matritsani R matritsa ko'rinishiga keltirish uchun ketma-ket $n-1$ marta o'xshash almashtirish yordamida A matritsaning satrlarini oxirgi satridan boshlab mos ravishda R matritsa satrlariga o'tkaziladi.

Faraz qilaylik, A matritsaning $a_{n,n-1}$ elementi noldan farqli bo'lsin va uni ajratilgan element deymiz. A matritsani o'ng tomondan

$$M_{n-1} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\dfrac{a_{n,1}}{a_{n,n-1}} & -\dfrac{a_{n,2}}{a_{n,n-1}} & \dots & -\dfrac{a_{n,n-2}}{a_{n,n-1}} & \dfrac{1}{a_{n,n-1}} & -\dfrac{a_{n,n}}{a_{n,n-1}} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

matritsaga ko'paytiramiz, natijada

$$AM_{n-1}=B=\begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n-1} & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n-1} & b_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ b_{n-1,1} & b_{n-1,2} & \dots & b_{n-1,n-1} & b_{n-1,n} \\ 0 & 0 & \dots & 1 & 0 \end{vmatrix}$$

hosil bo'ladi. Matritsalami ko'paytirish qoidasiga ko'ra, B matritsaning elementlari

$$b_{ij}=a_{ij}-a_{i,n-1}\cdot\frac{a_{nj}}{a_{n,n-1}},\quad j=1,2,\dots,n;\ j\neq n-1;$$

$$b_{i,n-1}=\frac{a_{i,n-1}}{a_{n,n-1}},\quad i=1,2,\dots,n.$$

formulalar yordamida aniqlanadi.

Hosil bo'lgan B matritsa A matritsaga o'xshash bo'lishi uchun chapdan M_{n-1}^{-1} matritsani B martitsaga ko'paytirish kerak:

$$M_{n-1}^{-1}AM_{n-1}=M_{n-1}^{-1}B.$$

Bevosita tekshirib ko'rish bilan M_{n-1}^{-1} quyidagi ko'rinishda bo'lishligiga ishonch hosil qilinadi:

$$M_{n-1}^{-1}=\begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{n,n-1} & a_{n,n} \\ 0 & 0 & \dots & 0 & 1 \end{vmatrix}$$

$C=M_{n-1}^{-1}B$ deb belgilaylik. M_{n-1}^{-1} B matritsaning oxirgi yo'lini o'zgartirmasligi yaqqol ko'rinib turibdi. Demak, C matritsa

$$C=\begin{vmatrix} c_{11} & c_{12} & \dots & c_{1n-1} & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n-1} & c_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ c_{n-1,1} & c_{n-1,2} & \dots & c_{n-1,n-1} & c_{n-1,n} \\ 0 & 0 & \dots & 1 & 0 \end{vmatrix}$$

ko'rinishida bo'ladi. Ko'paytirish amali B matritsaning faqat $(n-1)$ satrini o'zgartirishini anglash ham qiyin emas. Bu yerda

$$c_{ij} = b_{ij}, \quad i = 1,2,\ldots,n-2, \quad j = 1,2,\ldots,n$$

$$c_{n-1,j} = \sum_{k=1}^{n} a_{nk} b_{kj}, \quad j = 1,2,\ldots,n$$

hosil bo'lgan C matritsa A matritsaga o'xshash va uning oxirgi satri kerakli ko'rinishga keltirilgan. Shu bilan metodning bitta qadami bajarildi. Endi ajratilgan element $C_{n-1,n-2} \neq 0$ bo'lsin deb, C matritsaning $(n-1)$ satrini Frobenius formasiga keltirish uchun birinchi qadamdagi amallarni C martitsaning $(n-1)$ satri uchun bajarish kerak, ya'ni

$$D = M_{n-2}^{-1} C M_{n-2}$$

amallarni bajarish kerak. Bu yerda

$$M_{n-2} = \begin{bmatrix} 1 & \ldots & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\dfrac{c_{n-1,1}}{c_{n-1,n-1}} & \cdots & -\dfrac{c_{n-1,n-3}}{c_{n-1,n-2}} & \dfrac{1}{c_{n-1,n-2}} & -\dfrac{c_{n-1,n-1}}{c_{n-1,n-2}} & -\dfrac{c_{n,n}}{c_{n-1,n-2}} \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{n-2}^{-1} = \begin{bmatrix} 1 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-1,1} & \cdots & c_{n-1,n-2} & c_{n-1,n-1} & c_{n-1,n} \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

Shunday qilib, D matritsaning oxirgi ikkita satri Frobenius formasiga keltirilgan bo'ladi. Shu jarayon $n-1$ marta bajarilishi mumkin bo'lsa, A matritsa Frobenius normal formasiga keltirilgan bo'ladi, ya'ni

$$M_1^{-1}M_2^{-1}\cdots M_{n-1}^{-1}AM_{n-1}\cdots M_2M_1=P$$

Bundan foydalanib,

$$D(\lambda)=\det(A-\lambda E)=\det(P-\lambda E)=\lambda^n-p_1\lambda^{n-1}-p_2\lambda^{n-2}-p_3\lambda^{n-3}-\cdots-p_n$$

ni hosil qilamiz. $D(\lambda)=0$ tenglamani yechib, $\lambda_1,\lambda_2,\ldots,\lambda_n$ lar aniqlanadi.

Danilevskiy metodida ajratilgan element nolga teng bo'lsa, bu hol noregulyar hol deyiladi. Bu holda Danilevskiy metodi bilan almashtirish jarayonini davom ettirib bo'lmaydi.

Faraz qilaylik, A matritsani Frobenius ko'rinishiga keltirishda $(n-k)$ qadam bajarilgan bo'lib, quyidagi

$$D=\begin{vmatrix} d_{11} & d_{12} & \ldots & d_{1k} & \ldots & d_{1,n-1} & d_{1,n} \\ d_{21} & d_{22} & \ldots & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ d_{k1} & d_{k2} & \ldots & d_{kk} & \ldots & d_{k,n-1} & d_{k,n} \\ 0 & 0 & \ldots & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & \ldots & 0 & 0 \end{vmatrix}$$

matritsa hosil bo'lgan va $d_{k,k-1}=0$ bo'lsin. Bu yerda ikki hol bo'lishi mumkin:

1-hol: $d_{k,k-l}$ dan chapdagi biror element $d_{k,e}\neq 0$, $1\leq l\leq k-1$ bo'lsa, D matritsaning $(k-1)$ ustunini l-ustun bilan, shuningdek, $(k-1)$ yo'lini l-yo'l biln almashtiramiz. Hosil bo'lgan martitsa D matritsaga o'xshash bo'ladi va Danilevskiy metodini davom ettirish mumkin.

2-hol. $d_{ke}=0$, $l=1,2,\ldots,k-1$ bo'lsin. U holda D

$$D_1=\left|\begin{array}{cccc|cccc} d_{11} & d_{12} & \ldots & d_{1k-1} & d_{1k} & \ldots & d_{1n-1} & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{2k-1} & d_{2k} & \ldots & d_{2n-1} & d_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ d_{k-1,1} & d_{k-1,2} & \ldots & d_{k-1,k-1} & d_{k-1,k-1} & \ldots & d_{k-1,n-1} & d_{k-1,n} \\ \hline 0 & 0 & \ldots & 0 & d_{k,k} & \ldots & d_{k,n-1} & d_{k,n} \\ 0 & 0 & \ldots & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 0 \end{array}\right|$$

ko'rinishga ega bo'ladi. Demak, $\det(A-\lambda E)=\det(D_1-\lambda E)\det(D_2-\lambda E)$. D_2 matritsa Frobenius normal formasiga ega. Danilevskiy metodini D_1 matritsaga qo'llab6 uni Frobenius normal formasiga keltiriladi.

Endi xos vektorni topish masalasini ko'raylik. Faraz qilaylik, A matritsaning, ya'ni R matritsaning ham barcha xos sonlari topilgan bo'lsin. R matritsaning berilgan λ xos soniga mos $y=(y_1, y_2, \ldots, y_n)'$ xos vektorini topamiz. $Py=\lambda y$ bo'lganligi uchun $(P-\lambda E)y=0$ yoki

$$\begin{bmatrix} p_1-\lambda & p_2 & p_3 & \ldots & p_{n-1} & p_n \\ 1 & -\lambda & 0 & \ldots & 0 & 0 \\ 0 & 1 & -\lambda & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & -\lambda \end{bmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = 0.$$

Buni ochib yozaylik:

$$\begin{cases} (p_1-\lambda)y_1 + p_2 y_2 + \ldots + p_n y_n = 0, \\ y_1 - \lambda y_2 = 0, \\ y_2 - \lambda y_3 = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ y_{n-1} - \lambda y_n = 0. \end{cases} \qquad (1)$$

Bu sistemadan

$$y_{n-1} = \lambda y_n,$$
$$y_{n-2} = \lambda^2 y_n,$$
$$\dots\dots\dots$$
$$y_1 = \lambda^{n-1} y_n$$

ni topamiz. Xos vektor xossasiga ko'ra y_n=1 deb olish mumkin, u holda

$$y_1 = \lambda^{n-1},$$
$$y_2 = \lambda^{n-2},$$
$$\dots\dots\dots$$
$$y_{n-1} = \lambda, \qquad (2)$$
$$y_n = 1$$

ga ega bo'lamiz. (2) ni (1)ning birinchisiga qo'ysak, u

$$D(\lambda) = \lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - p_3\lambda^{n-3} - \dots - p_n = 0$$

ko'rinishga ega bo'ladi, bu esa hisoblash jarayonini nazorat qilishga xizmat qiladi.

$$P = M_1^{-1}M_2^{-1}\cdots M_{n-1}^{-1}AM_{n-1}\cdots M_2M_1,$$
$$M_1^{-1}M_2^{-1}\cdots M_{n-1}^{-1}AM_{n-1}\cdots M_2M_1y = \lambda y$$

tenglikni chapdan M_1 ga, so'ng M_2 ga va h.k., oxirida M_{n-1} ga ko'paytirsak,

$$x = M_{n-1}M_{n-2}\dots M_1 y$$

ekanligi hosil bo'ladi. Ma'lumki, M_1 matritsa y vektorning birinchi koordinatasini o'zgartiradi, M_2 esa M_1y vektorning ikkinchi koordinatasini o'zgartiradi. Shu jarayonni $n-1$ marta takrorlasak, x vektorning hamma koordinatalari hisoblangan bo'ladi. Bu yo'l bilan noregulyar holning ikkinchi variantida xos vektorni topib bo'lmaydi. Bunday holatda xos vektorni, misol uchun, Krilov metodi bilan topish ma'quldir.

6.4-§. Matritsaning moduli bo'yicha eng katta xos son va unga mos xos vektorini topish

Faraz qilaylik, A matritsaning barcha xos vektorlari chiziqli erkli bo'lsin. Bu holda A matritsa oddiy strukturaga ega deyiladi.

1-hol. A matritsaning xos qiymatlari quyidagi tengsizlikni qanoatlantirsin

$$|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n| \tag{1}$$

Biz λ_1 ning taqribiy qiymatini topish usulini ko'rsatamiz. Ixtiyoriy noldan farqli $y^{(0)}$ vektor olib, uni A matritsaning xos vektorlari $x^{(j)}$ lar bo'yicha yoyamiz:

$$y^{(0)} = b_1x^{(1)} + b_2x^{(2)} + \ldots + b_nx^{(n)},$$

bu yerda b_j lar o'zgarmas sonlar. $y^{(0)}$ vektor ustida A^k matritsa yordamida almashtirish bajaramiz:

$$y^{(k)} = A^k y^{(0)} = \sum_{j=1}^{n} b_j A^k x^{(j)} = \sum_{j=1}^{n} b_j \lambda_j^k x^{(j)}.$$

Bundan $Ax^{(j)} = \lambda_j x^{(j)}$ ligini e'tiborga olsak,

$$y^{(k)} = \sum_{j=1}^{n} b_j \lambda_j^k x^{(j)} \tag{2}$$

bo'ladi.

Endi n o'lchovli vektorlar fazosida $l_1, l_2, \ldots, l_n$ bazis olamiz. Bu bazisda $x^{(j)}$ vektorni yoyib yozamiz:

$$x^{(j)} = \sum_{i=1}^{n} x_{ij} l_i. \tag{3}$$

Endi (2) dan (3) ga asosan

$$y^{(k)} = \sum_{j=1}^{n} b_j \lambda_j^k \sum_{i=1}^{n} x_{ij} l_i$$

ni hosil qilamiz. Bunda yig'ish tartibini o'zgartirib,

$$y^{(k)} = \sum_{i=1}^{n} l_i \sum_{j=1}^{n} b_j \lambda_j^k x_{ij} \tag{4}$$

ga ega bo'lamiz. Ichki summa l_i ning koeffitsiyenti, demak u $y^{(k)}$ vektorning i-koordinatasidir. Bundan quyidagini yoza olamiz:

$$y_i^{(k)} = \sum_{j=1}^{n} b_j \lambda_j^k x_{ij} \tag{5}$$

Xuddi shuningdek,

$$y_i^{(k+1)} = \sum_{j=1}^{n} b_j \lambda_j^{k+1} x_{ij} \tag{6}$$

(6) ni (5) ga nisbatini olib,

$$\frac{y_i^{(k+1)}}{y_i^{(k)}} = \frac{b_1 x_{i1} \lambda_1^{k+1} + b_2 x_{i2} \lambda_2^{k+1} + \ldots + b_n x_{in} \lambda_n^{k+1}}{b_1 x_{i1} \lambda_1^k + b_2 x_{i2} \lambda_2^k + \ldots + b_n x_{in} \lambda_n^k} \tag{7}$$

ga ega bo'lamiz.

Endi $b_1 x_{i1} \neq 0$ deylik, bunga erishish uchun $y^{(0)}$ vektorni va $l_1, l_2, \ldots, l_n$ bazisni kerakli ravishda tanlash kerak.

(7) ni quyidagicha

$$\frac{y_i^{(k+1)}}{y_i^{(k)}} = \lambda_1 \frac{1 + \sum_{j=2}^{n} \frac{b_j x_{ij}}{b_1 x_{i1}} \left(\frac{\lambda_j}{\lambda_1} \right)^{k+1}}{1 + \sum_{j=2}^{n} \frac{b_j x_{ij}}{b_1 x_{i1}} \left(\frac{\lambda_j}{\lambda_1} \right)^{k}}$$

yozamiz. Bu yerdan $k \to \infty$ da (1) ga ko'ra

$$\lim_{k \to \infty} \frac{y_i^{(k+1)}}{y_i^{(k)}} = \lambda_1, \; i = 1, 2, \ldots, n$$

kelib chiqadi. Demak, yetarlicha katta k lar uchun

$$\lambda_1 \approx \frac{y_i^{(k+1)}}{y_i^{(k)}}, \quad i = 1, 2, \ldots, n \tag{8}$$

deb olishimiz mumkin.

Aniqlangan λ_1 ga mos xos vektor sifatida $u^{(k)}$ ni olish mumkin. Haqiqatan, (2) ga ko'ra

$$y^{(k)} = b_1 \lambda_1^k \left[x^{(1)} + \sum_{j=2}^{n} \frac{b_j}{b_1} \left(\frac{\lambda_j}{\lambda_1} \right)^k x^{(j)} \right]$$

bo'ladi. Yetarlicha katta k lar uchun

$$y^{(k)} \cong b_1\lambda_1^k x^{(1)}$$

taqribiy tenglikka ega bo'lamiz. $y^{(k)}$ xos vektor $x^{(1)}$ dan sonli ko'paytuvchiga farq qilyapti, demak, y λ_1 xos songa mos keladigan xos vektordir. U A matritsaning λ_1 xos soniga mos keluvchi $x^{(1)}$ xos vektorining yo'nalishiga yaqin bo'ladi. (8) ning o'ng tomoni i=1,2,…,n lar uchun berilgan aniqlikda bir xil bo'lishligi λ_1 va $x^{(1)}$ ga yaqinlashganlik darajasini anglatadi.

2-hol. A matritsa xos sonining moduli bo'yicha eng kattasi karrali bo'lsin, ya'ni $\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_s$. Quyidagi

$$|\lambda_1| > |\lambda_{s+1}| \geq |\lambda_{s+2}| \geq \cdots \geq |\lambda_n| \qquad (9)$$

tengsizlik bajarilsin. Bu holda (7) tenglik

$$\frac{y_i^{(k+1)}}{y_i^{(k)}} = \frac{(b_1x_{i1}+b_2x_{i2}+\ldots+b_sx_{is})\lambda_1^{k+1}+b_{s+1}x_{is+1}\lambda_{s+1}^{k+1}+\ldots+b_nx_{in}\lambda_n^{k+1}}{(b_1x_{i1}+b_2x_{i2}+\ldots+b_sx_{is})\lambda_1^{k}+b_{s+1}x_{is+1}\lambda_{s+1}^{k}+\ldots+b_nx_{in}\lambda_n^{k}} \qquad (10)$$

ko'rinishga ega bo'ladi. Bunda $b_1x_{i1} + b_2x_{i2} + \ldots + b_sx_{is} \neq 0$ deb faraz qilamiz va (10) ni quyidagi ko'rinishda yozamiz:

$$\frac{y_i^{(k+1)}}{y_i^{(k)}} = \lambda_1 \frac{1+\dfrac{1}{b_1x_{i1}+b_2x_{i2}+\cdots+b_sx_{is}}\sum\limits_{j=s+1}^{n} b_jx_{ij}\left(\dfrac{\lambda_j}{\lambda_1}\right)^{k+1}}{1+\dfrac{1}{b_1x_{i1}+b_2x_{i2}+\cdots+b_sx_{is}}\sum\limits_{j=s+1}^{n} b_jx_{ij}\left(\dfrac{\lambda_j}{\lambda_1}\right)^{k}}$$

Bundan $k \to \infty$ da (9)ga ko'ra

$$\lim_{k\to\infty}\frac{y_i^{(k+1)}}{y_i^{(k)}} = \lambda_1, \quad i = 1,2,\ldots,n$$

kelib chiqadi. Demak, yetarlicha katta k lar uchun

$$\lambda_1 \approx \frac{y_i^{(k+1)}}{y_i^{(k)}}, \quad i = 1,2,\ldots,n$$

deb olishimiz mumkin. Bu esa (8) taqribiy tenglikning o'zginasi, λ_1 ga mos vektor deb, 1-holdagidek $y^{(k)}$ ni olishimiz mumkin. Boshlang'ich $y^{(0)}$ vektorni boshqacha olsak, boshqa xos vektorni topish mumkin.

3-hol. A matritsaning xos sonlari quyidagi shartlarni qanoatlantirsin:

$$\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_r = -\lambda_{r+1} = -\lambda_{r+2} = \ldots = -\lambda_{r+p} \qquad (11)$$

va

$$|\lambda_1| = |\lambda_{r+p}| > |\lambda_{r+p+1}| \geq \cdots \geq |\lambda_n|. \qquad (12)$$

Bu holda yuqoridagi jarayondan foydalanilmaydi, chunki (5) quyidagi ko‘rinishga ega bo‘lib,

$$y_i^{(k)} = \sum_{j=1}^{r} b_j x_{ij} \lambda_1^k + \sum_{j=r+1}^{r+p} b_j x_{ij} (-1)^k \lambda_1^k + \sum_{j=r+p+1}^{n} b_j x_{ij} \lambda_j^k,$$

birinchi va ikkinchi summada λ_1 ning tartibi bir xil, lekin k ning o‘zgarishi bilan ikkinchi summa o‘z ishorasini o‘zgartiradi. Shuning uchun

$$\frac{y_i^{(k+1)}}{y_i^{(k)}}$$

nisbat $k \to \infty$ da limitga ega bo‘lmaydi. Bu holda $y_i^{(2k)}$ va $y_i^{(2k+2)}$ yoki $y_i^{(2k-1)}$ va $y_i^{(2k+1)}$ dan foydalanib, λ_1^2 ni topish mumkin:

$$\frac{y_i^{(2k+2)}}{y_i^{(2k)}} \approx \lambda_1, \quad i = 1,2,\ldots,n$$

yoki

$$\frac{y_i^{(2k+1)}}{y_i^{(2k-1)}} \approx \lambda_1, \quad i = 1,2,\ldots,n.$$

A matritsaning λ_1 va $-\lambda_1$ xos sonlariga mos keladigan vektorlari esa mos ravishda $y^{(k+1)} + \lambda_1 y^{(k)}$ va $y^{(k+1)} - \lambda_1 y^{(k)}$ bo‘ladi. Haqiqatan ham, misol uchun $y^{(k+1)} + \lambda_1 y^{(k)}$ vektorni (11) ni e’tiborga olgan holda (2) ga ko‘ra quyidagicha yozamiz:

$$y^{(k+1)} + \lambda_1 y^{(k)} = \lambda_1^{k+1} \sum_{j=1}^{r} b_j x^{(j)} + (-1)^{k+1} \lambda_1^{k+1} \sum_{j=r+1}^{r+p} b_j x^{(j)} + \sum_{j=r+p+1}^{n} b_j \lambda_j^{k+1} x^{(j)} +$$

$$+\lambda_1^{k+1} \sum_{j=1}^{r} b_j x^{(j)} + (-1)^{k+1} \lambda_1^{k} \sum_{j=r+1}^{r+p} b_j x^{(j)} + \lambda_1 \sum_{j=r+p+1}^{n} b_j \lambda_j^k x^{(j)} =$$

$$= 2\lambda_1^{k+1} \sum_{j=1}^{r} b_j x^{(j)} + \sum_{j=r+p+1}^{n} b_j \left(\lambda_1 + \lambda_j\right) \lambda_j^k x^{(j)}.$$

Bundan esa, (12)ni nazarga olib

$$y^{(k+1)}+\lambda_1 y^{(k)}=\lambda_1^{k+1}\left(\sum_{j=1}^{r}2b_jx^{(j)}+\sum_{j=r+p+1}^{n}\left(\lambda_1^{2}+\lambda_1\lambda_j\right)\left(\frac{\lambda_j}{\lambda_1}\right)^k b_jx^{(j)}\right)$$

ga ega bo'lamiz. Demak, yetarlicha katta k uchun

$$y^{(k+1)}+\lambda_1 y^{(k)}\cong\lambda_1^{k+1}\sum_{j=1}^{r}2b_jx^{(j)}$$

bo'ladi. Xuddi shuningdek,

$$y^{(k+1)}-\lambda_1 y^{(k)}\cong\left(-\lambda_1\right)^{k+1}\sum_{j=r+1}^{r+p}2b_jx^{(j)}$$

ekanligi ko'rsatiladi.

Agar r va p yoki ularning birortasi birdan katta bo'lsa, dastlabki vektor $y^{(0)}$ ni o'zgartirib boshqa xos vektorlarni topish mumkin.

4-hol. A matritsaning moduli bo'yicha eng katta sonlari kompleks yoki modullari bilan o'zaro yaqin bo'lgan holni ko'ramiz. Faraz qilaylik, λ_1 va λ_2 xos sonlar qo'shma kompleks sonlar bo'lib, quyidagi shartlar o'rinli bo'lsin:

$$|\lambda_1|=|\lambda_2|\geq|\lambda_3|\geq|\lambda_4|\geq\cdots\geq|\lambda_n|.$$

Quyidagi taqribiy tengliklar

$$\begin{cases} y^{(k)}\cong b_1\lambda_1^{k}x^{(1)}+b_2\lambda_2^{k}x^{(2)}, \\ y^{(k+1)}\cong b_1\lambda_1^{k+1}x^{(1)}+b_2\lambda_2^{k+1}x^{(2)}, \\ y^{(k+2)}\cong b_1\lambda_1^{k+2}x^{(1)}+b_2\lambda_2^{k+2}x^{(2)} \end{cases} \quad (13)$$

mavjudligiga ishonch hosil qilish qiyin emas. Bular orasida

$$y^{(k+2)}-\left(\lambda_1+\lambda_2\right)y^{(k+1)}+\lambda_1\lambda_2 y^{(k)}=0 \quad (14)$$

chiziqli bog'lanish o'rinlidir. Agar hisoblash jarayonida $y^{(k)}$, $y^{(k+1)}$, $y^{(k+2)}$ vektorlar orasida

$$y^{(k+2)}+py^{(k+1)}+qy^{(k)}=0 \quad (15)$$

chiziqli bog'lanish borligini ko'rsak, u holda λ_1 va λ_2 lar

$$u^2+pu+q=0 \quad (16)$$

kvadrat tenglamani qanoatlantiradi. (16) kvadrat tenglamaning koeffitsiyentlari quyidagi determinantlardan aniqlanadi:

$$\begin{vmatrix} 1 & y_i^{(k)} & y_j^{(k)} \\ U & y_i^{(k+1)} & y_j^{(k+1)} \\ U^2 & y_i^{(k+2)} & y_j^{(k+2)} \end{vmatrix} = 0, \quad i \neq j, \quad i,j = 1,2,\ldots,n \qquad (17)$$

Demak, (17) dan p va q topiladi, (16)dan esa λ_1 va λ_2 aniqlanadi, so'ngra (13) dan foydalanib xos vektorlar quyidagicha topiladi:

$$y^{(k+1)} - \lambda_1 y^{(k)} = b_2 \lambda_2{}^k (\lambda_2 - \lambda_1) x^{(1)},$$

$$y^{(k+1)} - \lambda_2 y^{(k)} = b_1 \lambda_1{}^k (\lambda_1 - \lambda_2) x^{(2)}.$$

Eslatma. Birinchi va ikkinchi hollarda $y^{(0)}$ vektorning iteratsiyalarini topish lozim edi. Shu jarayonni tezlashtirish uchun quyidagicha yo'l tutiladi:

$$A, A^2, A^4, A^8 \ldots A^{2^k}$$

matritsalar ketma-ketligini hosil qilamiz.

Ma'lumki,

$$\sum_{i=1}^{n} \lambda_i = SpA,$$

$$\sum_{i=1}^{n} \lambda_i{}^{2^k} = SpA^{2^k}$$

Bundan foydalanib, misol uchun 1-holda

$$\sum_{i=1}^{n} \lambda_i = SpA,$$

$$\sum_{i=1}^{n} \lambda_i{}^{2^k} = SpA^{2^k}.$$

lardan

$$\lambda_1 \cong \frac{SpA^{2^k+1}}{SpA^{2^k}}$$

ekanligi kelib chiqadi.

Bobga tegishli tayanch so'zlar: xos son, xos vektor, xos qiymatlarning to'liq muammosi, xarakteristik ko'phad, minimal ko'phad, Frobenius normal matritsasi.

Savollar va topshiriqlar

1. Matritsaning xos soni va xos vektori tushunchasi.
2. Matritsaning xarakteristik ko'phadi.
3. Xarakteristik ko'phad koeffitsiyentlari bilan matritsa bosh minorlari orasidagi bog'lanish.
4. Xos sonlarning barchasining yig'indisi hamda ko'paytmasi nimaga teng?
5. Xos qiymatlarning to'liq va qismiy muammolari.
6. Krilov usuli bilan xarakteristik ko'phad koeffitsiyentlarini topish.
7. Krilov usuli bilan xos vektorlarni topish.
8. Matritsaning minimal ko'phadining koeffitsiyentlarini topish qanday bajariladi?
9. Danilevskiy metodining asosiy g'oyasi.
10. Danilevskiy metodida xos vektorni topish.
11. Noregulyar hollar.
12. Noregulyar holning qaysi variantida xos vektorlarni Danilevskiy metodi bilan aniqlab bo'lmaydi?
13. Moduli bo'yicha eng katta xos son yagona bo'lganda uni va unga mos xos vektorni topish.
14. Moduli bo'yicha eng katta xos son karrali bo'lganda ($\lambda_1=\lambda_2=...=\lambda_s$, $|\lambda_1|>|\lambda_{s+1}|\geq\cdots\geq|\lambda_n|$ hol) uni va unga mos xos vektorni topish.
15. Xos sonlar uchun $\lambda_1=\lambda_2=\cdots=\lambda_r=-\lambda_{r+1}=-\lambda_{r+2}=\cdots=-\lambda_{r+p}$ va $|\lambda_1|=|\lambda_{r+p}|>|\lambda_{r+p+1}|\geq\cdots\geq|\lambda_n|$ shartlar o'rinli bo'lganda xos son va xos vektorni topish.

16. λ_1 va λ_2 qo'shma kompleks sonlar bo'lib $|\lambda_1|=|\lambda_2|>|\lambda_3|\geq|\lambda_4|\geq\cdots\geq|\lambda_n|$ shart o'rinli bo'lganda xos son va xos vektorni topish.

Misol 1.

$$A=\begin{pmatrix}1&2&4\\2&1&2\\4&2&1\end{pmatrix}$$

matritsaning xos qiymatlari va xos vektorlarini Krilov usuli bilan toping.

Yechish. Noldan farqli $y^{(0)}=\left(y_1^{(0)},y_2^{(0)},y_3^{(0)}\right)'$ vektor olib uning iteratsiyalari $y^{(k)}=Ay^{(k-1)}$, $(k=1,2,3)$ ni aniqlaymiz. $y^{(0)}=(1,0,0)'$ bo'lsin, u holda $y^{(1)}=Ay^{(0)}=(1,2,4)'$, $y^{(2)}=Ay^{(1)}=(21,12,12)'$, $y^{(3)}=Ay^{(2)}=(93,78,120)'$ bo'ladi. Endi $p_1y^{(2)}+p_2y^{(1)}+p_3y^{(0)}=y^{(3)}$ vektor tenglamani tuzamiz:

$$\begin{pmatrix}21\\12\\12\end{pmatrix}p_1+\begin{pmatrix}1\\2\\4\end{pmatrix}p_2+\begin{pmatrix}1\\0\\0\end{pmatrix}p_3=\begin{pmatrix}93\\78\\120\end{pmatrix}$$

yoki

$$\begin{cases}21p_1+p_2+p_3=93,\\12p_1+2p_2=78,\\12p_1+4p_2=120.\end{cases}$$

Bundan $p_1=3$, $p_2=21$, $p_3=9$ ekanligini topamiz. Berilgan matritsaning xarakteristik ko'phadi

$$D(\lambda)=\lambda^3-3\lambda^2-21\lambda-9$$

ko'rinishda bo'lar ekan. $D(\lambda)=0$ tenglamani yechamiz:

$$\lambda^3-3\lambda^2-21\lambda-9=0.$$

$\lambda=-3$ tenglamaning bitta ildizi ekanligini ko'rish qiyin emas. Qolgan ikkitasi $\lambda^2-6\lambda-3=0$ ning ildizlari. Shunday qilib, berilgan matritsaning xos sonlari $\lambda_1=-3, \lambda_2=3-\sqrt{12}, \lambda_3=3+\sqrt{12}$ ekan. Endi xos vektorlarni topishga o'tamiz.

λ_1 uchun:

$$\varphi_1(\lambda)=\frac{D(\lambda)}{\lambda-\lambda_1}=\frac{\lambda^3-3\lambda^2-21\lambda-9}{\lambda+3}=\lambda^2-6\lambda-3,$$

demak, $q_{11}=-6, q_{21}=-3$ ko'rinishda $\varphi_1(\lambda)=\lambda^2+q_{11}\lambda+q_{21}$ ekan. Xos vektor

$$c_1\varphi_1'(\lambda_1)x^{(1)}=y^{(2)}+q_{11}y^{(1)}+q_{21}y^{(0)}$$

ko'rinishda topilgan edi.

$$y^{(2)}+q_{11}y^{(1)}+q_{21}y^{(0)}=\begin{pmatrix}21\\12\\12\end{pmatrix}-6\cdot\begin{pmatrix}1\\2\\4\end{pmatrix}-3\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=$$

$$=\begin{pmatrix}21-6-3\\12-12+0\\12-24+0\end{pmatrix}=\begin{pmatrix}12\\0\\-12\end{pmatrix}$$

λ_2 uchun:

$$\varphi_2(\lambda)=\frac{D(\lambda)}{\lambda-\lambda_2}=\lambda^2-\sqrt{12}\lambda-3\left(3+\sqrt{12}\right),$$

$$q_{12}=-\sqrt{12},\quad q_{22}=-3\left(3+\sqrt{12}\right).$$

$$y^{(2)}+q_{12}y^{(1)}+q_{22}y^{(0)}=$$

$$=\begin{pmatrix}21\\12\\12\end{pmatrix}-\sqrt{12}\cdot\begin{pmatrix}1\\2\\4\end{pmatrix}-3\left(3+\sqrt{12}\right)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}12-4\sqrt{12}\\12-2\sqrt{12}\\12-4\sqrt{12}\end{pmatrix}$$

λ_3 uchun:

$$\varphi_3(\lambda)=\frac{D(\lambda)}{\lambda-\lambda_3}=\lambda^2+\sqrt{12}\lambda-3\left(3-\sqrt{12}\right),$$

$$q_{13}=\sqrt{12},\quad q_{23}=-3\left(3-\sqrt{12}\right).$$

$$y^{(2)}+q_{13}y^{(1)}+q_{23}y^{(0)}=$$

$$=\begin{pmatrix}21\\12\\12\end{pmatrix}+\sqrt{12}\cdot\begin{pmatrix}1\\2\\4\end{pmatrix}-3\left(3-\sqrt{12}\right)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}12+4\sqrt{12}\\12+2\sqrt{12}\\12+4\sqrt{12}\end{pmatrix}$$

Misol 2.
$$A=\begin{pmatrix}5 & 30 & -48\\3 & 14 & -24\\3 & 15 & -25\end{pmatrix}$$

matritsaning xos son va xos vektorlarini Krilov usuli bilan topilsin.

Yechish. $y^{(0)}=(1,0,0)'$ bo'lsin, u holda $y^{(1)}=(5;3;3)'$, $y^{(2)}=(-29;-15;-15)'$, $y^{(3)}=(125;63;63)'$ bo'ladi. Endi quyidagi

$$\begin{cases}-29p_1+5p_2+p_3=125,\\-15p_1+3p_2=63,\\-15p_1+3p_2=63.\end{cases}$$

sistemani tuzamiz. Ikkinchi va uchinchi tenglamalar bir xil, demak $y^{(0)}$, $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ vektorlar chiziqli bog'liq. Shuning uchun $y^{(0)}, y^{(1)}, y^{(2)}$ larga bog'liq

$$\begin{cases}5p_1+p_2=-29,\\3p_1=-15\end{cases}$$

sistemani tuzamiz. Bundan $p_1=-5$, $p_2=-4$ bo'ladi. $\lambda^2+5\lambda+4$ ko'phad A matritsaning minimal ko'phadidir. Undan $\lambda_1=-4$, $\lambda_2=-1$ ni topamiz. λ_3 ni topish uchun, bizga ma'lum

$$a_{11}+a_{22}+a_{33}=\lambda_1+\lambda_2+\lambda_3$$

munosabatdan foydalanib

$$5+14-25=-4-1+\lambda_3,\ \ \lambda_3=-1$$

ekanligini aniqlaymiz.

Xos vektorlar

$$x^{(i)}=\beta_{i1}y^{(0)}+\beta_{i2}y^{(1)},\ \ i=1,2$$

ko'rinishda izlanadi, bu yerda $\beta_{i1}=\lambda_i-p_1,\ \ \beta_{i2}=1,\ \ i=1,2$

$$x^{(1)}=\beta_{11}y^{(0)}+\beta_{12}y^{(1)}=(\lambda_1-p_1)y^{(0)}+1\cdot y^{(1)}=(-4+5)\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}5\\3\\3\end{pmatrix}=\begin{pmatrix}6\\3\\3\end{pmatrix}.$$

$$x^{(2)}=\beta_{21}y^{(0)}+\beta_{22}y^{(1)}=(\lambda_2-p_1)y^{(0)}+1\cdot y^{(1)}=(-1+5)\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}5\\3\\3\end{pmatrix}=\begin{pmatrix}9\\3\\3\end{pmatrix}.$$

$x^{(3)}$ ni topish uchun $y^{(0)}$ vektorni boshqacha tanlash kerak. Xususan, $y^{(0)}=(0;1;0)'$ desak $y^{(1)}=(30,14,15)'$ bo'ladi.

$$x^{(3)}=(\lambda_3-p_1)\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}+1\cdot\begin{pmatrix}30\\14\\15\end{pmatrix}=(-1+5)\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}30\\14\\15\end{pmatrix}=\begin{pmatrix}30\\18\\15\end{pmatrix}$$

$x^{(3)}=(30;18;15)'$ bo'lar ekan.

Misol 3.

$$A=\begin{pmatrix}1&2&4\\2&1&2\\4&2&1\end{pmatrix}$$

matritsaning barcha xos sonlari va xos vektorlarini, ya'ni xos qiymatlarning to'liq muammosini Danilevskiy metodi bilan aniqlang.

Yechish. $$M_2=\begin{pmatrix}1 & 0 & 0\\ -\frac{a_{31}}{a_{32}} & \frac{1}{a_{32}} & -\frac{a_{33}}{a_{31}}\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ -2 & \frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & 1\end{pmatrix}$$

Endi A matritsa ustida M_2 almashtirish bajarsak, matritsaning oxirgi yo'li normal Frobenius ko'rinishiga keladi:

$$A\cdot M_2=\begin{pmatrix}1 & 2 & 4\\ 2 & 1 & 2\\ 4 & 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -2 & \frac{1}{2} & -\frac{1}{2}\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}-3 & 1 & 3\\ 0 & \frac{1}{2} & \frac{3}{2}\\ 0 & 1 & 0\end{pmatrix}$$

$$M_2^{-1}=\begin{pmatrix}1 & 0 & 0\\ a_{31} & a_{32} & a_{33}\\ 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 4 & 2 & 1\\ 0 & 0 & 1\end{pmatrix}$$

Endi AM_2 matritsani chapdan M_2^{-1} ga ko'paytirsak, hosil bo'lgan matritsa A ga o'xshash bo'ladi:

$$M_2^{-1}\cdot A\cdot M_2=\begin{pmatrix}1 & 0 & 0\\ 4 & 2 & 1\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}-3 & 1 & 3\\ 0 & \frac{1}{2} & \frac{3}{2}\\ 0 & 1 & 0\end{pmatrix}=\begin{pmatrix}-3 & 1 & 3\\ -12 & 6 & 15\\ 0 & 1 & 0\end{pmatrix}=C.$$

Hosil bo'lgan C matritsa A ga o'xshash bo'ldi, uning ikkinchi yo'lini normal Frobenius ko'rinishiga keltiramiz:

$$M_1=\begin{pmatrix}\frac{1}{c_{21}} & -\frac{c_{22}}{c_{21}} & -\frac{c_{23}}{c_{21}}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}-\frac{1}{12} & \frac{1}{2} & \frac{5}{4}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$C\cdot M_1=\begin{pmatrix}-3 & 1 & 3\\ -12 & 6 & 15\\ 0 & 1 & 0\end{pmatrix}\cdot\begin{pmatrix}-\frac{1}{12} & \frac{1}{2} & \frac{5}{4}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

$$M_1^{-1}=\begin{pmatrix} c_{21} & c_{22} & c_{23} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} -12 & 6 & 15 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_1^{-1}\cdot C\cdot M_1=\begin{pmatrix} -12 & 6 & 15 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 3 & 12 & 9 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}=P$$

Hosil bo'lgan matritsa normal Frobenius ko'rinishida, uning xarakteristik tenglamasi $D(\lambda)=\lambda^3-3\lambda^2-21\lambda-9$ ko'rinishda bo'ladi. Krilov metodida ham u shu ko'rinishda edi, ya'ni uning nollari $\lambda_1=-3, \lambda_2=3-\sqrt{12}, \lambda_3=3+\sqrt{12}$ bo'lishi ravshan. P matritsaning xos vektorlari $y^{(i)}=(\lambda_i^2,\lambda_i,1)'$, $i=1,2,3$ edi. A ning xos vektorlari esa $x^{(i)}=M_2M_1y^{(i)}$, $i=1,2,3$ ko'rinishda bo'ladi:

$$x^{(1)}=M_2M_1y^{(1)}=M_2\cdot\begin{pmatrix} -\frac{1}{12} & \frac{1}{2} & \frac{5}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 9 \\ -3 \\ 1 \end{pmatrix}=M_2\cdot\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}=$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ -2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$x^{(1)}=(-1;\ 0;\ 1)'$ vektor Krilov usulida topilgan xos vektordan o'zgarmas ko'paytuvchi songa farq qilyapti. Xuddi shunday $x^{(2)}$ va $x^{(3)}$ larni aniqlashimiz mumkin.

Misol 4.

$$A=\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

matritsaning moduli bo'yicha eng katta xos sonini va unga mos xos vektorini verguldan keyin to'rt xona aniqlikda toping.

Yechish. Noldan farqli ixtiyoriy $y^{(0)}=\left(y_1^{(0)},y_2^{(0)},y_3^{(0)}\right)'$ vektor olib uning iteratsiyalari $y^{(k+1)}=Ay^{(k)}$, $k=1,2,....$ larni hosil qilamiz. $y^{(0)}=(1;\ 0;\ 0)'$ bo'lsin. U holda misol uchun $y^{(5)}=(683;341;341)'$ $y^{(6)}=(2731;\ 1365;1365)'$ bo'ladi va jarayonni shu yerda to'xtatamiz.

$\frac{y_j^{(6)}}{y_j^{(5)}}\approx\lambda_1$ edi. $\frac{y_1^{(6)}}{y_1^{(5)}}=\frac{2731}{683}\approx3,9985$, $\frac{y_2^{(6)}}{y_2^{(5)}}=\frac{1365}{341}\approx4,0029$, $\frac{y_3^{(6)}}{y_3^{(5)}}=\frac{1365}{341}\approx4,0029$.

Ko'rinib turibdiki, bu nisbatlar to'rt soni atrofida tebranyapti, demak natijalarni o'rta arifmetigini λ_1 deb olish mumkin.

$$\lambda_1\cong\frac{1}{3}(3,9985+8,0058)=4,0014.$$

λ_1 ga mos keluvchi xos vektor

$$y^{(5)}\cong b_1\lambda_1^5x^{(1)}$$

taqribiy tenglikdan topilar edi, ya'ni $y^{(5)}$ vektor xos vektor $x^{(1)}$ dan sonli ko'paytuvchiga farq qilyapti, demak $y^{(5)}$ λ_1 ga mos keladigan xos vektordir.

$(683;\ 341;341)'$ $-\lambda_1$ ga mos keluvchi xos vektor.

Misollar.

Vazifa 1. Krilov usuli bilan matritsaning xos sonlari va xos vektorlari topilsin. Xos sonlar to'rtta, xos vektorlar esa uchta ishonchli raqamlar bilan topilsin.

$$№1.\ A=\begin{pmatrix} 1 & 1{,}5 & 2{,}5 & 3{,}5 \\ 1{,}5 & 1 & 2 & 1{,}6 \\ 2{,}5 & 2 & 1 & 1{,}7 \\ 3{,}5 & 1{,}6 & 1{,}7 & 1 \end{pmatrix}.$$

$$№2.\ A=\begin{pmatrix} 1 & 1{,}2 & 2 & 0{,}5 \\ 1{,}2 & 1 & 0{,}4 & 1{,}2 \\ 2 & 0{,}4 & 2 & 1{,}5 \\ 0{,}5 & 1{,}2 & 1{,}5 & 1 \end{pmatrix}.$$

$$№3.\ A=\begin{pmatrix} 1 & 1{,}2 & 2 & 0{,}5 \\ 1{,}2 & 1 & 0{,}5 & 1 \\ 2 & 0{,}5 & 2 & 1{,}5 \\ 0{,}5 & 1 & 1{,}5 & 0{,}5 \end{pmatrix}.$$

$$№4.\ A=\begin{pmatrix} 2{,}5 & 1 & -0{,}5 & 2 \\ 1 & 2 & 1{,}2 & 0{,}4 \\ -0{,}5 & 1{,}2 & -1 & 1{,}5 \\ 2 & 0{,}4 & 1{,}5 & 1 \end{pmatrix}.$$

$$№5.\ A=\begin{pmatrix} 2 & 1 & 1{,}4 & 0{,}5 \\ 1 & 1 & 0{,}5 & 1 \\ 1{,}4 & 0{,}5 & 2 & 1{,}2 \\ 0{,}5 & 1 & 1{,}2 & 0{,}5 \end{pmatrix}.$$

$$№6.\ A=\begin{pmatrix} 2 & 1{,}2 & -1 & 1 \\ 1{,}2 & 0{,}5 & 2 & -1 \\ -1 & 2 & -1{,}5 & 0{,}2 \\ 1 & -1 & 0{,}2 & 1{,}5 \end{pmatrix}.$$

№7. $A=\begin{pmatrix} 2 & 1,5 & 3,5 & 4,5 \\ 1,5 & 2 & 2 & 1,6 \\ 3,5 & 2 & 2 & 1,7 \\ 4,5 & 1,6 & 1,7 & 2 \end{pmatrix}$.

№8. $A=\begin{pmatrix} 1 & 0,5 & 1,2 & -1 \\ 0,5 & 2 & -0,5 & 0 \\ 1,2 & -0,5 & -1 & 1,4 \\ -1 & 0 & 1,4 & 1 \end{pmatrix}$.

№9. $A=\begin{pmatrix} 1,2 & 0,5 & 2 & 1 \\ 0,5 & 1 & 0,8 & 2 \\ 2 & 0,8 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$.

№10. $A=\begin{pmatrix} 0,5 & 1,2 & 1 & 0,9 \\ 1,2 & 2 & 0,5 & 1,2 \\ 1 & 0,5 & 1 & 1 \\ 0,5 & 1,2 & 1 & 2,2 \end{pmatrix}$.

№11. $A=\begin{pmatrix} 1,6 & 1 & 1,4 & 1 \\ 1 & 1 & 0,5 & 2 \\ 1,4 & 0,5 & 2 & 1,2 \\ 1 & 2 & 1,2 & 0,5 \end{pmatrix}$.

№12. $A=\begin{pmatrix} 2,4 & 0,5 & 2 & 1 \\ 0,5 & 1 & 0,8 & 2 \\ 2 & 0,8 & 1 & 0,5 \\ 1 & 2 & 0,5 & 1,2 \end{pmatrix}$.

№13. $A=\begin{pmatrix} 0,5 & 1,2 & 2 & 1 \\ 1,2 & 2 & 0,5 & 1,2 \\ 2 & 0,5 & 1 & 0,5 \\ 1 & 1,2 & 0,5 & 1,6 \end{pmatrix}$.

$$№14.\ A=\begin{pmatrix} 1,8 & 1,6 & 1,7 & 1,8 \\ 1,6 & 2,8 & 1,5 & 1,3 \\ 1,7 & 1,5 & 3,8 & 1,4 \\ 1,8 & 1,3 & 1,4 & 4,8 \end{pmatrix}.$$

$$№15.\ A=\begin{pmatrix} 1 & 1,5 & 1,2 & 0,5 \\ 1,5 & 2 & 0,4 & 2 \\ 1,2 & 0,4 & 1,5 & 1,4 \\ 0,5 & 2 & 1,4 & 1,3 \end{pmatrix}.$$

Vazifa 2. Danilevskiy metodi bilan matritsaning xos sonlari va xos vektorlari topilsin. Xos sonlar to'rtta, xos vektorlar esa uchta ishonchli raqamlar bilan topilsin.

$$№1.\ A=\begin{pmatrix} 1,7 & 2,8 & 0,3 \\ 2,8 & 1,2 & 0,6 \\ 0,3 & 0,6 & 1,5 \end{pmatrix}. \qquad №2.\ A=\begin{pmatrix} 1,7 & 0,4 & 2,8 \\ 0,4 & 3,2 & 1,2 \\ 2,8 & 1,2 & 0,5 \end{pmatrix}.$$

$$№3.\ A=\begin{pmatrix} 2,3 & 1,4 & 0,6 \\ 1,4 & 1,7 & 0,5 \\ 0,6 & 0,5 & 1,3 \end{pmatrix}. \qquad №4.\ A=\begin{pmatrix} 2,3 & 3,5 & 1,4 \\ 3,5 & 0,4 & 0,6 \\ 1,4 & 0,6 & 1,3 \end{pmatrix}.$$

$$№5.\ A=\begin{pmatrix} 0,6 & 1,3 & 1,7 \\ 1,3 & 2,5 & 0,8 \\ 1,7 & 0,8 & 1,4 \end{pmatrix}. \qquad №6.\ A=\begin{pmatrix} 3,7 & 0,3 & 1,2 \\ 0,3 & 2,4 & 0,8 \\ 1,2 & 0,8 & 1,5 \end{pmatrix}.$$

$$№7.\ A=\begin{pmatrix} 3,2 & 0,5 & 1,2 \\ 0,5 & 1,4 & 2,3 \\ 1,2 & 2,3 & 0,6 \end{pmatrix}. \qquad №8.\ A=\begin{pmatrix} 4,1 & 0,4 & 1,3 \\ 0,4 & 2,2 & 1,7 \\ 1,3 & 1,7 & 0,5 \end{pmatrix}.$$

$$№9.\ A=\begin{pmatrix} 2,3 & 0,7 & 0,6 \\ 0,7 & 3,4 & 1,2 \\ 0,6 & 1,2 & 1,7 \end{pmatrix}. \qquad №10.\ A=\begin{pmatrix} 1,5 & 0,8 & 2,9 \\ 0,8 & 3,4 & 2,2 \\ 2,9 & 2,2 & 0,4 \end{pmatrix}.$$

№11. $A=\begin{pmatrix} 1,8 & 2,4 & 0,5 \\ 2,4 & 1,3 & 0,7 \\ 0,5 & 0,7 & 1,6 \end{pmatrix}$. №12. $A=\begin{pmatrix} 0,7 & 1,5 & 3,2 \\ 0,7 & 2,3 & 1,3 \\ 3,2 & 1,3 & 0,4 \end{pmatrix}$.

№13. $A=\begin{pmatrix} 2,4 & 3,5 & 0,7 \\ 3,5 & 1,2 & 0,4 \\ 0,7 & 0,4 & 1,3 \end{pmatrix}$. №14. $A=\begin{pmatrix} 2,3 & 1,7 & 0,8 \\ 1,7 & 0,5 & 1,2 \\ 0,8 & 1,2 & 1,9 \end{pmatrix}$.

№15. $A=\begin{pmatrix} 2,4 & 1,3 & 0,5 \\ 1,3 & 0,8 & 2,4 \\ 0,5 & 2,4 & 3,3 \end{pmatrix}$.

Vazifa 3. Moduli bo'yicha eng katta xos sonni to'rtta ishonchli raqam bilan toping. Unga mos xos vektorni esa uchta ishonchli raqam bilan aniqlang.

№1. $A=\begin{pmatrix} 2,1 & 1 & 1,1 \\ 1 & 2,6 & 1,1 \\ 1,1 & 1,1 & 3,1 \end{pmatrix}$. №2. $A=\begin{pmatrix} 2,4 & 1 & 1,4 \\ 1 & 2,9 & 1,4 \\ 1,4 & 1,4 & 3,4 \end{pmatrix}$.

№3. $A=\begin{pmatrix} 1,3 & 0,4 & 0,5 \\ 0,4 & 1,3 & 0,3 \\ 0,5 & 0,3 & 1,3 \end{pmatrix}$. №4. $A=\begin{pmatrix} 1,6 & 0,7 & 0,8 \\ 0,7 & 1,6 & 0,3 \\ 0,8 & 0,3 & 1,6 \end{pmatrix}$.

№5. $A=\begin{pmatrix} 2,2 & 1 & 1,2 \\ 1 & 2,7 & 1,2 \\ 1,2 & 1,2 & 3,2 \end{pmatrix}$. №6. $A=\begin{pmatrix} 2,5 & 1 & 1,5 \\ 1 & 3 & 1,5 \\ 1,5 & 1,5 & 3,5 \end{pmatrix}$.

№7. $A=\begin{pmatrix} 1,4 & 0,5 & 0,6 \\ 0,5 & 1,4 & 0,3 \\ 0,6 & 0,3 & 1,4 \end{pmatrix}$. №8. $A=\begin{pmatrix} 1,7 & 0,8 & 0,9 \\ 0,8 & 0,7 & 0,3 \\ 0,9 & 0,3 & 1,7 \end{pmatrix}$.

№9. $A=\begin{pmatrix} 2,3 & 1 & 1,3 \\ 1 & 2,8 & 1,3 \\ 1,3 & 1,3 & 3,3 \end{pmatrix}$.

№10. $A=\begin{pmatrix} 2,6 & 1 & 1,6 \\ 1 & 3,1 & 1,6 \\ 1,6 & 1,6 & 3,6 \end{pmatrix}$.

№11. $A=\begin{pmatrix} 3,5 & 1 & 2,5 \\ 1 & 4 & 2,5 \\ 2,5 & 2,5 & 4,5 \end{pmatrix}$.

№12. $A=\begin{pmatrix} 1,8 & 0,9 & 1 \\ 0,9 & 1,8 & 0,3 \\ 1 & 0,3 & 1,8 \end{pmatrix}$.

№13. $A=\begin{pmatrix} 1,5 & 0,6 & 0,7 \\ 0,6 & 1,5 & 0,3 \\ 0,7 & 0,3 & 1,5 \end{pmatrix}$.

№14. $A=\begin{pmatrix} 2,7 & 1 & 1,7 \\ 1 & 3,2 & 1,7 \\ 1,7 & 1,7 & 3,7 \end{pmatrix}$.

№15. $A=\begin{pmatrix} 1,4 & 1,2 & -1,3 \\ 1,2 & 0,9 & 0,4 \\ -1,3 & 0,4 & 0,8 \end{pmatrix}$.

VII BOB. ODDIY DIFFERENSIAL TENGLAMALAR UCHUN KOSHI MASALASINI TAQRIBIY YECHISH

Ilmiy va tadbiqiy masalalarda ko'pincha shunday oddiy differensial tenglamalar uchraydiki, ularning umumiy yechimini oshkor ko'rinishda ifodalash mumkin emas. Yechimi oshkor ko'rinishda topiladigan differensial tenglamalar sinfi ancha tor. Masalan, ko'rinishi soddagina bo'lgan

$$y' = x + x^2 + y^2$$

differensial tenglamaning umumiy yechimini elementar funksiyalar orqali ifodalab bo'lmaydi. Bu yechim ancha murakkab tarzda kasr tartibli Bessel funksiyalari orqali ifodalanadi. Ko'p hollarda yechimning hatto shunaqa ifodasini ham bilmaymiz. Shuning uchun ham bunday tenglamalarni u yoki bu taqribiy metod bilan yechishga to'g'ri keladi.

Taqribiy yechim analitik ko'rinishda yoki jadval shaklida izlanishiga qarab taqribiy metodlar *analitik* va *sonli metodlarga* bo'linadi.

Oddiy differensial tenglama uchun Koshi masalasi va chegaraviy masala qo'yiladi. Koshi masalasini yechish chegaraviy masalani yechishga nisbatan ancha yengildir. Shuning uchun ham ayrim hollarda chegaraviy masala Koshi masalasiga keltirib yechiladi. Analitik metodlarga ketma-ket yaqinlashish (Pikar metodi), darajali qatorga yoyish, Chapligin, Nyuton-Kantorovich, kichik parametrlar metodlari kiradi. Ayniqsa, keyingi paytlarda EHMlarning rivojlanishi bilan aniqlik tartibi yuqori bo'lgan sonli metodlarga e'tibor kuchaydi. Shu bilan bir qatorda analitik metodlar hozir ham o'z tatbiqini yo'qotgani yo'q. Masalan, Koshi masalasini ko'p qadamli ayirmali metodlar bilan yechishda jadvalning boshidagi qiymatlarni, odatda, analitik metodlar bilan topiladi.

7.1-§. Ketma-ket yaqinlashish usuli

Faraz qilaylik, $[x_0, X]$ oraliqda

$$y' = f(x,y), \tag{1}$$

$$y(x_0) = y_0 \tag{2}$$

Koshi masalasi berilgan bo'lsin. Bu masala yechimini quyidagicha

$$y_n(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]dt$$
$$y_0(t) = y_0, \quad n = 1,2,... \tag{3}$$

aniqlangan ketma-ketlik ko'rinishda izlaymiz. (1) ning o'ng tomoni ma'lum shartlarni qanoatlantirganda, (3) ketma-ketlikning limiti (1) (2) masalani qanoatlantirishi differensial tenglamalar kursida ko'rsatilgan. Bu metodning amaliy qiymati yuqori emas. Uning kamchiligi, har bir keyingi yaqinlashishni topishda integrallash amalini bajarish kerak bo'ladi. Bundan tashqari, integrallashni aniq bajarilmaydigan hollar uchraganda uni taqribiy ravishda hisoblash kerak bo'ladi, bu esa, o'z navbatida, ko'p hisoblashlarni talab etadi. Shuning uchun ham ketma-ket yaqinlashish metodi boshqa metodlarni qo'llayotganda yordamchi metod sifatida qo'llaniladi. Ketma-ket yaqinlashish metodini hech qanday qiyinchiliksiz differensial tenglamalar sistemasiga qo'yilgan Koshi masalasiga qo'llash mumkin.

7.2-§. Darajali qator metodi

Bizga

$$y' = f(x,y), \tag{1}$$

$$y(x_0) = y_0 \tag{2}$$

Koshi masalasi berilgan bo'lsin.

Faraz qilaylik, $f(x,y)$ funksiya (x_0, y_0) nuqtada analitik bo'lsin, ya'ni u shu nuqtaning biror atrofida Teylor qatoriga yoyilsin:

$$y(x)=\sum_{p=0}^{\infty}\frac{y^{(p)}(x_0)}{p!}(x-x_0)^p, \qquad (3)$$

bu yerda $y^{(p)}(x_0)$ larni murakkab funksiyani differensiallash qoidasiga ko'ra ketma-ket hisoblab topiladi va (1), (2) ning yechimini taqriban quyidagi

$$y(x)\cong y_n(x)=\sum_{p=0}^{n}\frac{y^{(p)}(x_0)}{p!}(x-x_0)^p \qquad (4)$$

ko'rinishda olinadi. Agar $|x-x_0|$ kichik bo'lmasa, umuman $n\to\infty$ da $y_n(x)$ ning limiti $y(x)$ ga yaqinlashmasligi ham mumkin yoki yaqinlashish juda sust bo'ladi. Bunday holda quyidagicha ish qilinadi.

Faraz qilaylik, taqribiy yechimning x_0+l nuqtadagi qiymatini topish lozim bo'lsin, bu yerda $l>0$ va ancha katta $[x_0, x_0+l]$ oraliqni x_i, $i=1,2,...,N-1$ nuqtalar yordamida N ta bo'lakka bo'lamiz, ya'ni

$$x_0<x_1<x_2<\ldots<x_n=x_0+l.$$

(4) formuladan foydalanib, taqribiy yechimning x_1 nuqtadagi qiymati $y(x_1)$ ni hisoblaymiz. Endi (4) ni x_1 nuqta uchun yozib, $y(x_2)$ ni hisoblaymiz. x_2 nuqtani boshlang'ich nuqta deb, $y(x_3)$ ni hissoblaymiz, bu jarayonni $x_N=x_0+l$ nuqtaga yetguncha bajaramiz. Agar $\max\limits_{1\le i\le N}|x_i-x_{i-1}|$ yetarlicha kichik bo'lsa, (4) formulani yuqorida ko'rsatilgan algoritm bo'yicha qo'llash maqsadga muvofiqdir. Bunday algoritm juda ko'p hisoblashni taqozo etadi. Shuning uchun darajali qator metodini, odatda, x_0 ga yaqin nuqtalarda taqribiy yechimni topish uchun qo'llaniladi.

Shuni alohida ta'kidlash lozimki, (4) ni shundayligicha qo'llasak, darajali qator metodi analitik taqribiy metoddir, (4) ni qadamma-qadam x_i, $i=1,2,...,N-1$ nuqtalardagi yechimi qiymatini yuqorida ko'rsatilgan algoritm asosida qo'llasak, unda u sonli metodlar qatoriga o'tadi.

Bu metodni yuqori tartibli differensial tenglamaga hamda differensial tenglamalar sistemasiga qo‘yilgan Koshi masalasini yechishga qo‘llash mumkin.

7.3-§. Ayirmali metodlar

Faraz qilaylik, bizga quyidagi Koshi masalasi berilgan bo‘lsin

$$y' = f(x,y), \tag{1}$$

$$y(x_0) = y_0 \,. \tag{2}$$

Bu masalani $[x_0, X]$ oraliqda yechish talab etilgan bo‘lsin. Berilgan kesmani N ta teng bo‘lakka bo‘lamiz:

$$x_i = x_0 + ih, \quad i = 1,2,...,N, \quad N = \frac{X - x_0}{h}, \; h > 0.$$

(1), (2) Koshi masalasini yechish

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \tag{3}$$

integral tenglamani yechish bilan teng kuchlidir. Biroq (3)da integral ostida noma’lum funksiya qatnashishi masalani murakkablashtiradi. Ketma-ket ikkita nuqtadagi yechimning orasida quyidagi munosabat o‘rinlidir:

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} y'(x)\,dx. \tag{4}$$

$y(x_{n+1})$ ni topish uchun $y(x_n)$ va $y'(x) = f(x, y(x))$ ni $[x_n, x_{n+1}]$ oraliqda bilish kerak. $y'(x)$ ning taqribiy qiymatlari faqat $x_0, x_1, \ldots, x_n$ nuqtalardagina bizga ma’lum. Shuning uchun (4)dagi integralni taqribiy hisoblashga to‘g‘ri keladi. Bu integralning qanday hisoblanishiga qarab berilgan Koshi masalasini yechadigan taqribiy u yoki bu metod hosil bo‘ladi.

7.3.1. Adams ekstrapolyatsion metodi

Faraz qilaylik, (1), (2) Koshi masalasining yechimi $k+1$ ta $[x_n, x_{n+1}]$ oraliqdan chapda yotgan $x_{n-k}, x_{n-k+1}, \dots, x_n$ nuqtalarda ma'lum bo'lsin. U holda $y'(x)$ ning ham bu nuqtalardagi qiymatlari ma'lum bo'ladi.

(4) da $x = x_n + uh$ almashtirish bajarsak, u

$$y(x_{n+1}) = y_n + h\int_0^1 y'(x_n + uh)du \tag{5}$$

ko'rinishda bo'ladi. $y'(x_n + uh)$ funksiyani uning $k+1$ ta qiymatlaridan foydalanib, Nyutonning ikkinchi interpolyatsion ko'phadi bilan ekstrapolyatsiya o'tkazamiz, ya'ni

$$\begin{aligned} y'(x_n + uh) = y'(x_n) + \frac{u}{1!}\Delta y'(x_{n-1}) + \frac{u(u+1)}{2!}\Delta^2 y'(x_{n-2}) + \dots + \\ + \frac{u(u+1)\cdots(u+k-1)}{k!}\Delta^k y'(x_{n-k}) + ᴂ_k(u), \end{aligned} \tag{6}$$

bu yerda

$$ᴂ_n(u) = h^{k+1}\frac{u(u+1)\cdots(u+k)}{(k+1)!}y^{(k+2)}(\xi),$$

$$x_{n-k} < \xi = \xi(u) < x_{n+1}; \quad 0 \le u \le 1.$$

(6) ni (5) ga qo'yib integrallash amallarini bajarsak,

$$\begin{aligned} y_{n+1} = y(x_n) + h\Big[y'(x_n) + \frac{1}{2}\Delta y'(x_{n-1}) + \frac{5}{12}\Delta^2 y'(x_{n-2}) + \\ + \frac{3}{8}\Delta^3 y'(x_{n-3}) + \dots + C_k\Delta^k y'(x_{n-k})\Big] + R_k, \end{aligned} \tag{7}$$

ga ega bo'lamiz. Bunda

$$C_k = \int_0^1 \frac{u(u+1)\cdots(u+k)}{k!}du, \tag{8}$$

$$R_k = h\int_0^1 ᴂ_k(u)du = h^{k+2}\int_0^1 \frac{u(u+1)\cdots(u+k)}{(k+1)!}y^{(k+2)}(\xi)du.$$

R_k ning ifodasidagi integralda $u(u+1)\cdots(u+k)$ $[0,1]$ da ishora saqlaydi, demak, o'rta qiymat haqidagi teoremaga asosan va $y^{(k+2)}(x)$ ni uzluksiz deb, qoldiq hadni

$$R_k = h^{k+2}C_{k+1}y^{(k+2)}(\overline{\xi}),\ \ x_{n-k} \le \overline{\xi} \le x_{n+1} \tag{9}$$

ko'rinishda ifodalash mumkin. Uni baholasak

$$|R_k| \le h^{k+2}\ C_{k+1} \max_{[x_0, X]} \left|y^{(k+2)}(x)\right| \tag{10}$$

ga ega bo'lamiz. Agar $h>0$ yetarlicha kichik bo'lib, Koshi masalasining yechimi esa $(k+2)$-tartibli uzluksiz hosilaga ega bo'lsa, $R_k = o(h^{k+2})$ kattalikni e'tiborga olmasak ham bo'ladi. (7) da R_k ni tashlab yuborsak, *Adams ekstrapolyatsion metodi* deb nomlanuvchi

$$\begin{aligned} y_{n+1} = y_n + h\Big[y'_n + \frac{1}{2}\Delta y'_{n-1} + \frac{5}{12}\Delta^2 y'_{n-2} + \frac{3}{8}\Delta^3 y'_{n-3} + \frac{251}{720}\Delta^4 y'_{n-4} + \\ + \frac{95}{288}\Delta^5 y'_{n-5} + \frac{19087}{60480}\Delta^6 y'_{n-6} + \frac{5275}{17280}\Delta^7 y'_{n-7} + \cdots + C_k\Delta^k y'_{n-k} \Big] \end{aligned} \tag{11}$$

formulaga ega bo'lamiz.

(11) formulada yuqori tartibli chekli ayirma x_{n-k} nuqtada hisoblanganligi uchun hisoblash jarayonini $n=k$ dan boshlash mumkin, ya'ni $x_0, x_1, \ldots, x_k$ nuqtalardagi yechimning qiymatlari (jadvalning boshlang'ich qismi) topilgan bo'lishini taqozo etadi.

(11) da $k=0$ desak, hisoblashning eng sodda, ya'ni jadvalning boshlang'ich qismini qurish talab etilmaydigan

$$y_{n+1} = y_n + h\,f(x_n, y_n) \tag{12}$$

qoida hosil bo'ladi. Buni Eyler metodi deyiladi. Uning oddiy geometrik ma'nosidan uni siniq chiziqlar metodi deb ham ataladi. (12) qoidaning xatoligi h^2 tartibga egaligi (10) dan ko'rinib turibdi va uning aniqligi ancha past.

Eyler metodidan aniqligi yuqori bo'lgan va (11) ko'rinishga ega bo'lgan boshqa formulalar jadvalning boshlang'ich qismi $y_0, y_1, \ldots, y_k$ larni qurishni talab etadi. Adams ekstrapolyatsion metodining xatoligi bahosida noma'lum yechimning yuqori tartibli hosilasi modulining maksimumi qatnashishi uni amaliyotda qo'llash imkonini kamaytiradi. Amaliyotda k va h larni tanlashda quyidagiga amal qilinadi.

Hisoblashlarda qatnashuvchi eng so'nggi chekli ayirma tanlangan aniqlik chegarasida o'zgarmas bo'lishiga erishish kerak. Buni quyidagicha izohlash mumkin: (11) formulada birinchi tashlab yuboriladigan had berilgan aniqlikdagi hisoblash natijasiga ta'sir qilmasligi kerak.

k ning o'sishi bilan (11) formula hadlarining moduli bo'yicha kamayishi $h<1$ bo'lganda, asosan chekli ayirmalar modulining kamayuvchanligi evaziga bo'ladi, ya'ni

$$\Delta^k y'_{n-k} = o(h^k).$$

Adams ekstrapolyatsion metodini birinchi tartibli oddiy differensial tenglamalar sistemasiga qo'yilgan Koshi masalasini yechishga qiyinchiliksiz qo'llash mumkin. Buni quyidagi masalada ko'rsatamiz:

$$\begin{cases} y' = f(x,y,z), & y(x_0) = y_0, \\ z' = g(x,y,z), & z(x_0) = z_0. \end{cases}$$

Bitta tenglama holidagiga o'xshash ko'rinishda $y(x)$ va $z(x)$ funksiyalar uchun (7) formula tipidagi tenglikni yozish mumkin:

$$y(x_{n+1}) = y(x_n) + h\left[y'(x_n) + \frac{1}{2}\Delta y'(x_{n-1}) + \cdots + C_k\Delta^k y'(x_{n-k})\right] + R_k,$$

$$z(x_{n+1}) = z(x_n) + h\left[z'(x_n) + \frac{1}{2}\Delta z'(x_{n-1}) + \cdots + C_k\Delta^k z'(x_{n-k})\right] + R_k^1.$$

R_k, R_k^1 larni e'tiborga olmasak, bulardan (11) ga o'xshash

$$y_{n+1} = y_n + h\left[y'_n + \frac{1}{2}\Delta y'_{n-1} + \cdots + C_k\Delta^k y'_{n-k}\right],$$

$$z_{n+1} = z_n + h\left[z'_n + \frac{1}{2}\Delta z'_{n-1} + \cdots + C_k\Delta^k z'_{n-k}\right]$$

hisoblash formulasiga ega bo'lamiz.

7.3.2. Adams interpolyatsion metodi

Bu metodda ham yechimning taqribiy qiymatini y_n gacha aniqlangan deb, yechimning taqribiy qiymati $y_{n+1} \cong y(x_{n+1})$ ni topish masalasi qaraladi. Yechimning x_{n+1} nuqtadagi qiymatini aniqlash uchun 7.3-§ dagi (4) formuladan foydalanimiz, lekin $y'(x)$ ni $[x_n, x_{n+1}]$ oraliqda interpolyatsiyalash uchun jadvalning boshlang'ich qismidagi uning qiymatlari va x_{n+1} nuqtadagi qiymatini ham ishtirok ettiramiz, ya'ni interpolyatsiyalash amali o'z ma'nosini saqlashini ta'minlaymiz. Shu bilan Adams metodlarining farqi izohlanadi.

Faraz qilaylik, (1), (2) Koshi masalasining yechimlari $x_{n-k+1}, x_{n-k+2}, \ldots, x_{n+1}$ da aniqlangan bo'lsin. (4) formulada $x = x_{n+1} + uh$ almashtirish o'tkazsak, u quyidagi

$$y(x_{n+1}) = y_n + h\int_{-1}^{0} y'(x_n + uh)du$$

ko'rinishga keladi va $y'(x_n + uh)$ ni Nyutonning ikkinchi interpolyatsion ko'phadiga almashtiramiz:

$$\begin{aligned} y'(x_n + uh) &= y'(x_{n+1}) + \frac{u}{1!}\Delta y'(x_n) + \frac{u(u+1)}{2!}\Delta^2 y'(x_{n-1}) + \cdots + \\ &+ \frac{u(u+1)\cdots(u+k-1)}{k!}\Delta^k y'(x_{n-k+1}) + r_k(u), \end{aligned} \tag{13}$$

bu yerda

$$r_n(u) = h^{k+1}\frac{u(u+1)\cdots(u+k)}{(k+1)!}y^{(k+2)}(\xi),$$

$$x_{n-k+1} < \xi = \xi(u) < x_{n+1}; \quad -1 \le u \le 0.$$

U holda

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + h\Big[y'(x_{n+1}) - \frac{1}{2}\Delta y'(x_n) - \frac{1}{12}\Delta^2 y'(x_{n-1}) - \\ &- \frac{1}{24}\Delta^3 y'(x_{n-2}) + \cdots + C_k\Delta^k y'(x_{n-k+1})\Big] + R_k. \end{aligned} \tag{14}$$

Bu yerda

$$C_k = \int_{-1}^{0} \frac{u(u+1)\cdots(u+k-1)}{k!} du,$$

$$R_k = h^{k+2} \int_{-1}^{0} \frac{u(u+1)\cdots(u+k)}{(k+1)!} y^{(k+2)}(\xi) du.$$

(14) da qoldiq hadni tashlab yuborsak,

$$y_{n+1} = y_n + h\Big[y'_{n+1} - \frac{1}{2}\Delta y'_n - \frac{1}{12}\Delta^2 y'_{n-1} - \frac{1}{24}\Delta^3 y'_{n-2} - \frac{19}{720}\Delta^4 y'_{n-3} - \\ - \frac{3}{160}\Delta^5 y'_{n-4} - \frac{863}{60480}\Delta^6 y'_{n-5} - \frac{275}{24195}\Delta^7 y'_{n-6} + \cdots + C_k \Delta^k y'_{n-k+1} \Big] \quad (15)$$

ko'rinishga ega va Adams interpolyatsion metodi deb nomlanuvchi algoritmga ega bo'lamiz. (15) formula bo'yicha hisoblash jarayoni to'g'ridan-to'g'ri o'tkazilmaydi, chunki u

$$y_{n+1} = \varphi(y_{n+1}) \quad (16)$$

ko'rinishdagi tenglamadir, bu yerda

$$\varphi(y_{n+1}) = h\left(1 + \sum_{i=1}^{k} C_i\right) f(x_{n+1}, y_{n+1}) + \\ + F(x_n, x_{n-1}, \ldots x_{n-k+1}, y_n, y_{n-1}, \ldots, y_{n-k+1})$$

ko'rinishga ega bo'lib, $F(x_n, x_{n-1}, \ldots x_{n-k+1}, y_n, y_{n-1}, \ldots, y_{n-k+1})$ funksiya argumentlariga nisbatan ma'lum funksiyadir. (16) tenglama iteratsiya metodi bilan yechiladi, chunki u iteratsiya metodini qo'llashdagi kanonik ko'rinishga ega. Agar $\frac{\partial f(x,y)}{\partial y}$ yechimning ma'lum bir atrofida uzluksiz bo'lsa, u holda boshlang'ich yaqinlashish yaxshi tanlangan bo'lsa yetarlicha kichik h uchun iteratsion jarayonning yaqinlashishini ta'minlash mumkin. Odatda, h shunday tanlanadiki, berilgan aniqlikka erishish uchun bitta yoki ikkita iteratsiya yetarli bo'lishi kerak. Shuni alohida ta'kidlaymizki, boshlang'ich yaqinlashish deb, (7) formula bilan topilgan qiymat maqsadga muvofiqdir. Amaliyotda Adamsning (7) va (15) formulalari hisoblash jarayonida ketma-ket navbati bilan ishlatiladi. Shuning uchun (7), (15) formulalar prediktor-korrektor deb ham ataladi.

Adams interpolyatsion metodini ham birinchi tartibli oddiy differensial tenglamalar sistemasiga qo'yilgan Koshi masalasini yechishda hech qanday qiyinchiliksiz qo'llash mumkin.

Ko'pincha hisoblash jarayonini osonlashtirish uchun (7) va (15) formulalarda ishtirok etuvchi chekli ayirmalar funksiyaning qiymatlari orqali ifodasiga almashtiriladi va k ning berilgan qiymatlarida Adams ekstrapolyatsion hamda interpolyatsion formulalari mos ravishda quyidagi ko'rinishda bo'ladi:

$$k=0,\quad y_{n+1}=y_n+hf(x_n,y_n),$$
$$y_{n+1}=y_n+hf(x_{n+1},y_{n+1}),$$

$$k=1,\quad y_{n+1}=y_n+h\left[\frac{3}{2}f_n-f_{n-1}\right],$$
$$y_{n+1}=y_n+h\left[\frac{1}{2}f_{n+1}+\frac{1}{2}f_n\right],$$

$$k=2,\quad y_{n+1}=y_n+\frac{h}{12}\left[23f_n-16f_{n-1}+5f_{n-2}\right],$$
$$y_{n+1}=y_n+\frac{h}{12}\left[5f_{n+1}+8f_n-f_{n-1}\right],$$

$$k=3,\quad y_{n+1}=y_n+\frac{h}{24}\left[55f_n-59f_{n-1}+37f_{n-2}-9f_{n-3}\right],$$
$$y_{n+1}=y_n+\frac{h}{24}\left[9f_{n+1}+19f_n-5f_{n-1}+f_{n-2}\right],$$

Yechimning x_{n+1} nuqtadagi qiymatini topishda undan farqli va bittadan ko'p bo'lgan nuqtadagi yechimning qiymatlari ishtirokida topiladigan metodlar *ko'p qadamli metodlar* deyiladi. Xususan, Adams metodlari ko'p qadamli metodlardir.

7.3.3. Eyler va Eyler-Koshi usullari

Yuqoridagi (7) formulada $k=0$ bo'lgan holda hisoblash jarayoni

$$y_{n+1}=y_n+hf_n \tag{16}$$

formula bilan tashkil etiladi. Buni Eyler taklif qilgan, shuning uchun bunday hisoblash jarayoni *Eyler metodi* deb yuritiladi.

Eyler usulining quyidagicha modifikatsiyasi mavjud.

Avval quyidagi yaqinlashish

$$y_{n+1/2}=y_n+\frac{h}{2}f_n=y_n+\frac{h}{2}f(x_n,y_n)$$

hisoblanadi, so'ng

$$y_{n+1}=y_n+hf_{n+1/2} \tag{17}$$

formula yordamida x_{n+1} nuqtadagi yechimning taqribiy qiymati topiladi. Bu yerda

$$y_{n+1/2}=y\left(x_n+\frac{h}{2}\right),\quad f_{n+1/2}=f\left(x_n+\frac{h}{2},y_n+\frac{1}{2}\right).$$

(17) ifoda *Eyler usulining modifikatsiyasidir.*

Eyler-Koshining modifikatsiyalangan metodi quyidagicha aniqlanadi.

Avval $\qquad \bar{y}_{n+1}=y_n+hf_n$

hisoblanadi, so'ng x_{n+1} nuqtadagi yechimning taqribiy qiymati

$$y_{n+1}=y_n+\frac{h}{2}\left(f_n+\bar{f}_{n+1}\right) \tag{18}$$

formula bilan aniqlanadi. Bu yerda

$$\bar{f}_{n+1}=f(x_{n+1},\bar{y}_{n+1}).$$

Agar (1), (2) masalaning taqribiy yechimini $y_{n+1}^{(0)}=y_n+hf(x_n,y_n)$ deb, uni quyidagi

$$y_{n+1}^{(k+1)}=y_k+\frac{h}{2}\left(f(x_n,y_n)+f(x_{n+1},y_{n+1}^{(k)})\right),\quad k=0,1,... \tag{19}$$

iteratsion jarayon bilan berilgan aniqlik miqdorida topilsa, (19) ifoda *Eyler-Koshining iteratsion metodi* deyiladi.

7.3.4. Runge-Kutta metodi

(1), (2) masalani yechish uchun (5) ni qulaylik uchun indekssiz quyidagicha

$$y(x+h) = y(x) + h\int_0^1 f(x+uh, y(x+uh))\,du \qquad (20)$$

ko'rinishda yozib olamiz.

Bu yerdagi integralni kvadratur formulalar yordamida taqribiy hisoblash mumkin emas, chunki integral ostida noma'lum funksiya qatnashyapti. Uni quyidagicha hisoblaymiz. Buning uchun uch guruhdan iborat

$$\alpha_2, \alpha_3, \ldots, \alpha_r; \qquad (\alpha)$$

$$\begin{array}{l} \beta_{21}, \\ \beta_{31}, \beta_{32}, \\ \ldots\ldots\ldots\ldots \\ \beta_{r1}, \beta_{r2}, \ldots, \beta_{rr-1} \end{array} \qquad (\beta)$$

$$p_1,\ p_2,\ \ldots,\ p_r \qquad (p)$$

hozircha noma'lum parametrlarni kiritamiz va bular yordamida (20) dagi integralni kvadratur summaga o'xshash chekli summaga almashtiramiz:

$$y(x+h) - y(x) \cong \sum_{i=1}^{r} p_i K_i \qquad (21)$$

bu yerda

$$K_1 = h\,f(x,y),$$

$$K_2 = h\,f(x+\alpha_2 h, y+\beta_{21} K_1),$$

$$K_3 = h\,f(x+\alpha_3 h, y+\beta_{31} K_1 + \beta_{32} K_2),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$K_r = h\,f(x+\alpha_r h, y+\beta_{r1} K_1 + \beta_{r2} K_2 + \cdots + \beta_{rr-1} K_{r-1}).$$

Agar (α) va (β) guruh parametrlar tanlangan bo'lsa, $K_1, K_2, \ldots, K_r$ miqdorlar ketma-ket hisoblanadi.

(21) taqribiy tenglikning xatoligi

$$\varphi_r(h) = \Delta y - \sum_{i=1}^{r} p_i K_i$$

bo'lsin. (1) ning o'ng tomoni yetarlicha silliq bo'lsa, u holda $\varphi_r(h)$ funksiya ham yetarlicha tartibgacha uzluksiz hosilalarga ega bo'ladi. Demak, quyidagi Makloren formulasini yoza olamiz:

$$\varphi_r(h) = \sum_{j=0}^{s} \frac{h^j}{j!} \varphi_r^{(j)}(0) + \frac{h^{s+1}}{(s+1)!} \varphi_r^{(s+1)}(\theta h), \quad 0 < \theta < 1. \qquad (22)$$

Agar (α), (β), (p) guruh parametrlarini tanlash evaziga $\varphi_r^{(j)}(0) = 0$, $j = 0,1,\ldots,s$ va $\varphi_r^{(s+1)}(0) \neq 0$ bo'lishini ta'minlasak, (21) formulaning xatoligi

$$\varphi_r(h) = \frac{h^{s+1}}{(s+1)!} \varphi_r^{(s+1)}(\theta h)$$

ko'rinishda bo'lib, uning qadami h ga nisbatan tartibi $s+1$ ga teng. Odatda, s ni Runge-Kutta metodining aniqlik darajasi deyiladi. Noma'lum (α), (β), (p) guruh elementlarini quyidagi talablardan topamiz. (21) ning chap va o'ng tomonini h ning darajalari bo'yicha yoyilmasida ixtiyoriy $f(x,y)$ va h uchun imkoni boricha h ning yuqori darajasigacha hadlar bir xil bo'lishini talab etamiz.

Boshqacha aytganda,

$$\varphi_r(h) = y(x+h) - y(x) - \sum_{i=1}^{r} p_i K_i(h)$$

funksiya

$$\varphi_r(0) = \varphi_r^{(1)}(0) = \cdots = \varphi_r^{(s)}(0) = 0, \quad \varphi_r^{(s+1)}(0) \neq 0$$

xossalarga ega bo'lib, (α), (β), (p) guruh elementlari shunday tanlanishi kerakki, ixtiyoriy h va $f(x,y)$ uchun s mumkin qadar katta bo'lsin.

Umumiy holda (α), (β), (p) guruh parametrlarni aniqlaydigan tenglamalar sistemasini yozish ancha murakkab bo‘lganligi uchun Runge-Kutta metodi bilan bir qadamli qoidalarning r ning to‘rtta qiymatidagisini topish masalasini ko‘ramiz.

$r = 1$ bo‘lsin, unda (21)

$$y(x+h)-y(x)\cong p_1\, h\, f(x,y)$$

ko‘rinishda bo‘ladi. Bundan

$$\varphi_1(h) = y(x+h)-y(x)-\, p_1\, h\, f(x,y)$$

ekanligini ko‘ramiz. Bundan hosilalar olamiz

$$\varphi_1'(h) = y'(x+h)-p_1\, f(x,y).$$

$$\varphi_1''(h) = y'(x+h)\ .$$

$\varphi_1''(0) = y''(x)$ miqdor p_1 parametrga bog‘liq emas va p_1 ni tanlash hisobiga nolga aylanmaydi. $\varphi_1'(0) = y'(x)-p_1\, f(x,y) = (1-p_1)f(x,y)$ miqdor ixtiyoriy $f(x,y)$ uchun $p_1 = 1$ bo‘lganda nolga teng bo‘ladi.

Demak, (21) $r = 1$ da $p_1 = 1$ bo‘lsa

$$y(x+h)-y(x)\cong p_1\, h\, f(x,y)$$

ko‘rinishda bo‘lib, uning xatoligi esa (22) ga asosan

$$\varphi_1(h) = \frac{h^2}{2}\varphi_1''(\theta h) = \frac{h^2}{2}y''(x+\theta h), \quad 0<\theta<1$$

bo‘ladi. Birinchi tartibli Runge-Kutta metodi Eyler metodi bilan ustma-ust tushar ekan.

Endi $r = 2$ bo‘lsin. (21) formula

$$y(x+h)-y(x)\cong p_1 K_1 + p_2 K_2 = h\, p_1 f(x,y) + h\, p_2 f(x+\alpha_2 h, y+\beta_{21} f(x,y))$$

ko‘rinishda bo‘ladi.

$p_1, p_2, \alpha_2, \beta_{21}$ parametrlarni aniqlash uchun $y(x+h)-y(x)$ va $p_1 K_1 + p_2 K_2$ larni h ning darajalari bo‘yicha yoyilmasini topamiz:

$$y(x+h)-y(x)=\frac{h}{1!}y'(x)+\frac{h^2}{2!}y''(x)+\frac{h^3}{3!}y'''(x)+o(h^4)=$$
$$=hf+\frac{h^2}{2!}\left(f_x+f\cdot f_y\right)+\frac{h^3}{3!}\left(f_{xx}+2f\cdot f_{xy}+f_y\left(f_x+f\cdot f_y\right)\right)+o(h^4) \tag{23}$$

$$p_1K_1+p_2K_2 = hp_1f(x,y)+hp_2f\left(x+\alpha_2h, y+\beta_{21}hf(x,y)\right)=$$
$$=h(p_1+p_2)+h^2p_2\left(\alpha_2f_x+\beta_{21}f\cdot f_y\right)+ \tag{24}$$
$$+\frac{h^3}{2}p_2\left(\alpha_2^2f_{xx}+2\alpha_2\beta_{21}f\cdot f_{xy}+\beta_{21}^2f^2\cdot f_{yy}\right)+o(h^4).$$

(23) va (24) yoyilmalarda ixtiyoriy $f(x,y)$ uchun hadlar h ning iloji boricha yuqori darajalarigacha ustma-ust tushsin, deb talab qilamiz. Bu ikki yoyilmani taqqoslasak, $p_1, p_2, \alpha_2, \beta_{21}$ larni topish uchun quyidagiga ega bo'lamiz:

$$hf \ : \ p_1+p_2=1,$$
$$h^2f_x \ : \ p_2\alpha_2=\frac{1}{2},$$
$$h^2f\cdot f_y \ : \ p_2\beta_{21}=\frac{1}{2}.$$

Yuqoridagi (23), (24) yoyilmalarning ko'rinishidan ma'lumki, $r=2$ da kiritilgan $p_1, p_2, \alpha_2, \beta_{21}$ parametrlarni tanlash evaziga ixtiyoriy $f(x,y)$ uchun h^3 li hadlarning bir xil bo'lmasligini ko'rish mumkin. Demak, yuqoridagi $p_1, p_2, \alpha_2, \beta_{21}$ larga bog'liq uchta tenglamadan $p_2 \neq 0$ deb

$$p_1=1-p_2, \alpha_2=\beta_{21}=\frac{1}{2p_2}$$

larni aniqlaymiz. Odatda, p_2 ni shunday tanlanadiki, hosil bo'ladigan formula hisoblashlarda qulaylikka ega bo'lishi kerak.

Misol uchun $p_2=\frac{1}{2}$ bo'lsin, unda $p_1=\frac{1}{2}$, $\alpha_2=\beta_{21}=1$ bo'ladi va quyidagi ikkinchi tartibli

$$y(x+h)=y(x)+\frac{h}{2}\left(f(x,y)+f(x+h,\, y+K_1)\right)$$

formula hosil bo'ladi.

Agar $p_2=1$ desak, unda $p_1=0$, $\alpha_2=\beta_{21}=\frac{1}{2}$ bo'ladi va

$$y(x+h)=y(x)+h\,f\left(x+\frac{h}{2},\, y+\frac{h}{2}f(x,y)\right)$$

ikkinchi tartibli hisoblash formulasiga ega bo'lamiz.

$r=3$ bo'lganda kiritilgan $\alpha_2, \alpha_3, p_1, p_2, p_3, \beta_{21}, \beta_{31}, \beta_{32}$ parametrlar uchun bajarilishi kerak bo'lgan shartlarni keltirish bilan chegaralanamiz, chunki (23) va (24) tipidagi yoyilmalar $r=3$ da bundan ham katta ko'rinishda bo'ladi. Ular quyidagicha

$$p_1+p_2+p_3=1,$$

$$p_2\alpha_2+p_3\alpha_3=\frac{1}{2},$$

$$p_2\alpha_2^2+p_3\alpha_3^2=\frac{1}{3},$$

$$p_2\alpha_2\beta_{32}=\frac{1}{6},$$

$$\beta_{31}+\beta_{32}=\alpha_3,$$

$$\beta_{21}=\alpha_2$$

ko'rinishga ega. Bu tenglamalar sistemasini yechimlaridan biri quyidagilar:

$$\alpha_2=\frac{1}{2},\ \ \alpha_3=1,\ \ \beta_{21}=\frac{1}{2},\ \ \beta_{31}=-1,\ \ \beta_{32}=2,$$

$$p_1=\frac{1}{6},\ \ p_2=\frac{2}{3},\ \ p_3=\frac{1}{6}.$$

Bu parametrlar bilan aniqlangan uchinchi tartibli Runge-Kutta metodi

$$y(x+h)=y(x)+\frac{1}{6}\left(K_1+4K_2+K_3\right)$$

ko'rinishda bo'lib, bu yerda

$$K_1 = hf(x, y), \ K_2 = hf\left(x + \frac{h}{2}, y + \frac{1}{2}K_1\right),$$
$$K_3 = hf\left(x + h, y - K_1 + 2K_2\right).$$

Bundan boshqa yana bir formulani keltiramiz:

$$y(x+h) = y(x) + \frac{1}{4}\left(K_1 + 3K_3\right)$$
$$K_1 = hf(x,y), \ K_2 = hf\left(x + \frac{h}{3}, y + \frac{1}{3}K_1\right),$$
$$K_3 = hf\left(x + \frac{2}{3}h, y + \frac{2}{3}K_2\right).$$

Endi $r = 4$ bo'lganda eng ko'p ishlatiladigan to'rtinchi tartibli metodni keltiramiz:

$$y(x+h) = y(x) + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right),$$

bu yerda

$$K_1 = hf(x,y), \ K_2 = hf\left(x + \frac{h}{2}, y + \frac{1}{2}K_1\right),$$
$$K_3 = hf\left(x + \frac{h}{2}, y + \frac{1}{2}K_2\right), \quad K_4 = hf\left(x + h, y + K_3\right).$$

Yuqorida chiqarilgan birinchi, ikkinchi, uchinchi va to'rtinchi tartibli metodlarni birinchi tartibli differensial tenglamalar sistemasi uchun Koshi masalasini taqribiy yechishda qo'llash mumkin. Masalan, quyidagi Koshi masalasi berilgan bo'lsin:

$$y_i' = f_i(x, y_1, y_2, \ldots, y_n),$$
$$y_i(x_0) = y_{i0}, \quad i = 1, 2, \ldots, n$$

$r = 1$ da hisoblash jarayoni quyidagicha amalga oshiriladi:

$$y_i(x+h) = y_i(x) + hf_i(x, y_1, y_2, \ldots, y_n), \ i = 1, 2, \ldots, n.$$

$r = 2$ da esa formulalar

$$y_i(x+h) = y_i(x) + \frac{1}{2}\left(K_{i1} + K_{i2}\right),$$
$$K_{i1} = hf(x, y_1, y_2, \ldots, y_n),$$
$$K_{i2} = hf(x + h, y_1 + K_{11}, y_2 + K_{21}, \ldots, y_n + K_{n1}), \ i = 1, 2, \ldots, n$$

ko'rinishga ega bo'ladi.

Runge-Kutta metodi bilan qurilgan metodlarda yechimning bitta nuqtadagi qiymati ishtirok etishi bois ular bir qadamli metodlar deyiladi.

7.3.5. Runge-Kutta metodi xatoligining bosh hadi va uni baholashda Runge qoidasi

Tartibi S ga teng Runge-Kutta metodi bilan Koshi masalasini yechishda har bir qadamdagi xatolik miqdori

$$\frac{h^{S+1}}{(S+1)!}\cdot\varphi_r^{(S+1)}(\theta h),\ 0<\theta<1$$

ga teng bo'lib, tartibi h^{S+1} edi. Agar differensial tenglamaning o'ng tomoni ancha murakkab bo'lsa, bu xatolikni baholash amaliyotda ancha qiyinchiliklarni tug'diradi. Metodning jami xatoligi tartibi esa h^S ga teng bo'ladi [8], ya'ni

$$y(x_n)=y_n+\rho(x_n)h^S+o(h^{S+1}). \qquad (1)$$

Agar $\rho(x_n)\neq0$ bo'lsa, yetarlicha kichik h uchun $\rho(x_n)h^S$ miqdor metod xatoligiga ancha yaxshi yaqinlashgan bo'ladi. Bu miqdor, odatda, xatolikning bosh hadi deb ataladi. Xatolikning bosh hadi xatolikni to'liq xarakterlamaydi, lekin uning miqdori metod xatoligi miqdorini yaxshi tasavvur etishga imkon beradi.

(1) formula yordamida xatolikning bosh hadini aposterior baholash qoidasi mavjud, uni *Runge qoidasi* deb ataladi.

Faraz qilaylik, $[x_0,X]$ oraliqning ξ nuqtasida tartibi S ga teng metod bilan $h=h_1$ va $h=h_2$ qadamlarda yechimning y_{h_1} va y_{h_2} taqribiy qiymatlari topilgan bo'lsin.

Agar h_1 va h_2 yetarlicha kichik bo'lsa, unda (1) ga asosan

$$y(\xi)-y_{h_1}\cong\rho(\xi)h_1^S,$$

$$y(\xi)-y_{h_2}\cong\rho(\xi)h_2^S$$

taqribiy tengliklarning xatoliklari ancha kichik bo'lishligini kutish mumkin. Bundan

$$\rho(\xi)\cong\frac{y_{h_1}-y_{h_2}}{h_2^S-h_1^S} \tag{2}$$

ligini aniqlaymiz. Demak, hisoblashlarni $h=h_1$, $h=h_2$ qadam bilan ξ nuqtagacha bajarsak, xatolikning bosh hadining taqribiy qiymatlarini ikkala holda ham aniqlagan bo'lamiz:

$$h_1^S\frac{y_{h_1}-y_{h_2}}{h_2^S-h_1^S},\quad h_2^S\frac{y_{h_1}-y_{h_2}}{h_2^S-h_1^S},$$

Agar bu miqdorlar berilgan aniqlik ε dan katta bo'lsa, quyidagi taqribiy tenglik

$$\varepsilon\approx|\rho(\xi)|h_\varepsilon^S\approx\left|\frac{y_{h_1}-y_{h_2}}{h_2^S-h_1^S}\right|h_\varepsilon^S$$

dan tegishli shartlarni bajaradigan qadamni ko'rsatish mumkin:

$$h_\varepsilon\approx\sqrt[S]{\varepsilon\left|\frac{h_2^S-h_1^S}{y_{h_1}-y_{h_2}}\right|}.$$

Agar $h_1=h$, $h_2=2h$ bo'lsa, (2) formula

$$\rho(\xi)=\frac{y_h-y_{2h}}{h^2(2^S-1)}$$

ko'rinishda bo'lib, metod xatoligi uchun

$$y-y_h\approx\frac{y_h-y_{2h}}{2^S-1} \tag{3}$$

taqribiy ifodaga ega bo'lamiz. Bu formula asimptotik xarakterga ega bo'lib, kafolatlangan bahoni bermaydi. Shunga qaramasdan, u amaliy nuqtayi nazardan ko'pincha yaxshi natijalarni beradi.

Shu narsani ta'kidlash lozimki, (3) formulada yaxlitlash xatoligi inobatga olinmagan. Shuning uchun, hisoblash jarayonida yaxlitlash xatoligi salmoqli bo'lsa, xatolikning bosh hadini baholashda (3) ni qo'llab bo'lmaydi.

Agar (3) ni quyidagicha

$$y\approx y_h+\frac{y_h-y_{2h}}{2^S-1} \tag{4}$$

yozsak, (4) ning o'ng tomoni yechimning aniq qiymati y ga, y_h va y_{2h} dan ko'ra ancha yaqinroq bo'lishligiga umid qilish mumkin.

Bobga tegishli tayanch so'zlar: bir qadamli metod, ko'p qadamli metod, Runge qoidasi.

Savollar va topshiriqlar

1. Koshi masalasini taqribiy yechishning usullari.
2. Ketma-ket yaqinlashish usuli.
3. Darajali qator metodi.
4. Ayirmali usullardan Adams ekstrapolyatsion metodi.
5. Adams interpolyatsion metodi.
6. Birinchi tartibli oddiy differensial tenglamalar sistemasi uchun Adams metodlari.
7. Eyler va Eyler–Koshi usullari.
8. Eyler–Koshining metodi.
9. Runge–Kutta metodi.
10. $r=1$, $r=2$ bo'lgandagi hisoblash formulalarini chiqaring.
11. $r=3$, $r=4$ bo'lgandagi hisoblash formulalarini yozing.
12. Xatolikni baholashda Runge qoidasi.

Misol 1.

$$y'=x+y\,, \tag{1}$$

$$y(0)=1 \tag{2}$$

Koshi masalasining $[0;0,4]$ oraliqda $h=0,1$ qadam bilan Eyler usulida yechimini toping. Taqribiy yechim va aniq yechim orasidagi farqni hisoblang. Analitik yechim $y(x)=2e^x-x-1$.

Yechish. Differensial tenglamaning o'ng tomoni $f(x,y)=x+y$;

boshlang'ich ma'lumotlar: $x_0=0$, $y_0=1$;

yechim izlanyotgan $[a,b]$ oraliq: $a=0$, $b=0,4$;

oraliqning bo'linish soni: $n=4$, $h=0,1$.

Quyidagi

$$x_{i+1}=x_i+0,1;\quad \Delta y_i=0,1(x_i+y_i);\quad y_{i+1}=y_i+\Delta y_i\quad (i=0,1,2,3)$$

rekurrent formulalardan foydalanib ketma-ket quyidagilarni topamiz:

1- qadamda ($i=0$): $x_1=0,1;\quad y_1=1+0,1(0+1)=1,1,$

2- qadamda ($i=1$): $x_2=0,2;\quad y_2=1,1+0,1(0,1+1,1)=1,22,$

3- qadamda ($i=2$): $x_3=0,3;\quad y_3=1,22+0,1(0,2+1,22)=1,362,$

4- qadamda ($i=3$): $x_4=0,4;\quad y_4=1,362+0,1(0,3+1,362)=1,5282.$

$d_i=\left|y(x_i)-y_i\right|$ deb belgilaymiz va hisoblashlarimiz natijalarini jadval ko'rinishida keltiramiz:

i	x_i	y_i	$y(x_i)$	d_i
1	0,1	1,1	1,110342	0,010342
2	0,2	1,22	1,242806	0,022806
3	0,3	1,362	1,399718	0,037718
4	0,4	1,5282	1,583649	0,055449

Misol 2.

$$y'=x+y \qquad (1)$$

$$y(0)=1 \qquad (2)$$

Koshi masalasini $[0;0,4]$ oraliqda $h=0,1$ qadam bilan Eyler--Koshi usuli bilan yeching.

Yechish. $x_0=0,\ \ y_0=1$ deb, Eyler–Koshi metodini indeks i uchun quyidagicha yozamiz:

$$x_{i+1}=x_i+h,\quad y_{i+1}=y_i+\Delta y_i\quad (i=0,1,2,3),$$

$$\Delta y_i=\frac{1}{2}\left(K_i^{(1)}+K_i^{(2)}\right),\quad K_i^{(1)}=h\cdot f(x_i,y_i),\quad K_i^{(2)}=h\cdot f\left(x_i+h,y_i+K_i^{(1)}\right).$$

Yechilyotgan masala uchun

$i=0;\quad x_1=x_0+h=0+0,1;\quad K_0^{(1)}=h(x_0+y_0)=0,1\cdot(0+1)=0,1;$

$K_0^{(2)}=h(x_0+h+y_0+K_0^{(1)})=0,1\cdot(0+0,1+1+0,1)=0,12;$

$\Delta y_0=\frac{1}{2}\left(K_0^{(1)}+K_0^{(2)}\right)=\frac{1}{2}(0,1+0,12)=0,11;$

$y_1=y_0+\Delta y_0=1+0,11=1,11.$

Xuddi shunday hisoblashlarni qolgan qadamlar uchun bajaramiz, xatolik $d_i=\left|y(x_i)-y_i\right|$ va hisoblashlar natijasini quyidagi jadvalda keltiramiz:

i	x_i	$K_{i-1}^{(1)}$	$K_{i-1}^{(2)}$	Δy_{i-1}	y_i	$y(x_i)$	d_i
1	0,1	0,1	0,12	0,11	1,11	1,110342	0,000342
2	0,2	0,121	0,1431	0,13205	1,24205	1,242805	0,000756
3	0,3	0,144205	0,168626	0,156415	1,398466	1,399718	0,001252
4	0,4	0,169847	0,196831	0,183339	1,581804	1,583649	0,001845

Topilgan taqribiy yechim xatoligi $|y_4 - y(x_4)| \approx 0,002$ dan ortmaydi.

Misol 3. Yuqoridagi misolni to'rtinchi tartibli Runge–Kutta usuli bilan yeching.

Yechish.

$$x_{i+1} = x_i + h,\ \ K_i^{(1)} = hf(x_i, y_i),\ K_i^{(2)} = hf\left(x_i + \frac{h}{2}, y_i + \frac{K_i^{(1)}}{2}\right),$$

$$K_i^{(3)} = hf\left(x_i + \frac{h}{2}, y_i + \frac{K_i^{(2)}}{2}\right),\quad K_i^{(4)} = hf\left(x_i + h, y_i + K_i^{(3)}\right),$$

$$\Delta y_i = \frac{1}{6}\left(K_i^{(1)} + 2K_i^{(2)} + 2K_i^{(3)} + K_i^{(4)}\right),\quad y_{i+1} = y_i + \Delta y_i\ \ (i = 0,1,2,3).$$

$x_0 = 0,\ \ y_0 = 1$ deb y_1 ni topamiz:

$$i=0;\quad x_1 = x_0 + h = 0 + 0,1;\quad K_0^{(1)} = h(x_0 + y_0) = 0,1(0+1) = 0,1;$$

$$K_0^{(2)} = h\left(x_0 + \frac{1}{2}h + \frac{1}{2}K_0^{(1)}\right) = 0,1(0 + 0,05 + 1 + 0,05) = 0,11;$$

$$K_0^{(3)} = h\left(x_0 + \frac{1}{2}h + y_0 + \frac{1}{2}K_0^{(2)}\right) = 0,1(0 + 0,05 + 1 + 0,055) = 0,1105;$$

$$K_0^{(4)} = h\left(x_0 + h,\ y_0 + K_0^{(3)}\right) = 0,1(0 + 0,1 + 1 + 0,1105) = 0,12105;$$

$$\Delta y_0 = \frac{1}{6}\left(K_0^{(1)} + 2K_0^{(2)} + 2K_0^{(3)} + K_0^{(4)}\right) = \frac{1}{6}(0,1 + 2\cdot 0,11 + 2\cdot 0,1105 + 0,12105) \cong$$

$$\cong 0,110342;$$

$$y_1 = y_0 + \Delta y_0 \approx 1 + 0,110342 = 1,110342.$$

Shunday hisoblashlarni keyingi qadamlar uchun ham bajarib hisoblash natijalarini esa quyidagi jadvalda keltiramiz:

i	x_i	$K_{i-1}^{(1)}$	$K_{i-1}^{(2)}$	$K_{i-1}^{(3)}$	$K_{i-1}^{(4)}$	Δy_{i-1}	y_i
1	0,1	0,100000	0,110000	0,110500	0,121050	0,110342	1,110342
2	0,2	0,121034	0,132086	0,132638	0,144298	0,132468	1,242805
3	0,3	0,144281	0,156495	0,157105	0,169991	0,156912	1,399717
4	0,4	0,169972	0,183470	0,184145	0,198386	0,183931	1,583648

Topilgan taqribiy yechim xatoligi 0,000001 dan ortmaydi.

Taqqoslash uchun uchta metod bilan yechilgan bitta masalani tugun nuqtalardagi yechimning taqribiy qiymatlarini va aniq yechimning qiymatlarini quyidagi jadvalda keltiramiz.

i	x_i	Usullar bo'yicha topilgan y_i qiymati			Aniq yechimi $y(x_i)=2e^x-x-1$
		Eyler	Eyler–Koshi	Runge–Kutta	
1	0,1	1,1	1,11	1,110342	1,110342
2	0,2	1,22	1,24205	1,242805	1,242805
3	0,3	1,362	1,398466	1,399717	1,399718
4	0,4	1,5282	1,581804	1,583648	1,583649

Misollar. Koshi masalasini $[0;1]$ oraliqda $h=0,1$ qadam bilan yechimini Eyler, Eyler–Koshi va Runge–Kutta usullarida toping.

№ 1. $y' = x + y^2, y(0) = 0,5.$ № 2. $y' = 2x + 0,1y^2, y(0) = 0,2.$

№ 3. $y' = 2x + y^2, y(0) = 0,3.$ № 4. $y' = x^2 + xy, y(0) = 0,2.$

№ 5. $y' = 0,2x + y^2, y(0) = 0,1.$ № 6. $y' = x^2 + y, y(0) = 0,4.$

№ 7. $y' = x^2 + 2y, y(0) = 0,1.$ № 8. $y' = xy + y^2, y(0) = 0,6.$

№ 9. $y' = x^2 + y^2, y(0) = 0,7.$ № 10. $y' = x^2 + 0,2y^2, y(0) = 0,2.$

№ 11. $y' = 0,3x + y^2, y(0) = 0,4.$ № 12. $y' = 0,1x + 0,2y^2, y(0) = 0,3.$

№ 13. $y' = x + 0,3y^2, y(0) = 0,3.$ № 14. $y' = 2x^2 + xy, y(0) = 0,5.$

№ 15. $y' = 0,1x^2 + 2xy,\ y(0) = 0,8.$ № 16. $y' = x^2 + 0,2xy,\ y(0) = 0,6.$

№ 17. $y' = 3x^2 + 0,1xy,\ y(0) = 0,2.$ № 18. $y' = x^2 + 3xy,\ y(0) = 0,3.$

№ 19. $y' = x^2 + 0,1y^2,\ y(0) = 0,7.$ № 20. $y' = 2x^2 + 3y^2,\ y(0) = 0,2.$

№ 21. $y' = 0,2x^2 + y^2, y(0) = 0,8.$ № 22. $y' = 0,3x^2 + 0,1y^2, y(0) = 0,3.$

№ 23. $y' = xy + 0,1y^2,\ y(0) = 0,5.$ № 24. $y' = 0,2xy + y^2,\ y(0) = 0,4.$

№ 25. $y' = 0,1xy + 0,3y^2,\ y(0) = 0,2.$ № 26. $y' = 0,3xy + y^2,\ y(0) = 0,6.$

№ 27. $y' = xy + 0,2y^2,\ y(0) = 0,7.$ № 28. $y' = 0,1x^2 + 2y^2,\ y(0) = 0,2.$

№ 29. $y' = 3x + 0,1y^2,\ y(0) = 0,4$ № 30. $y' = 0,2x + 3y^2,\ y(0) = 0,2.$

VIII BOB. ODDIY DIFFERENSIAL TENGLAMALAR UCHUN CHEGARAVIY MASALALARNI TAQRIBIY YECHISH

8.1-§. Masalaning qo'yilishi

Faraz qilaylik, quyidagi n-tartibli oddiy differensial tenglama

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \qquad (1)$$

berilgan bo'lsin. Uning $y = y(x)$ yechimini $[a,b]$ oraliqda topish talab qilinsin. Bu oraliqda k ta x_i $(i = 1,2,\ldots,k)$ nuqtalar olamiz:

$$a \le x_1 < x_2 < x_3 < \cdots < x_k \le b.$$

Yechim $y(x)$ va uning $(n-1)$-tartibgacha hosilalarini x_i $(i = 1,2,\ldots,k)$ nuqtalardagi qiymatlaridan qandaydir qoidaga binoan tuzilgan quyidagicha tenglamalar berilgan bo'lsin:

$$Y_j\left(y(x_1), y'(x_1), \ldots, y^{(n-1)}(x_1), \ldots, y(x_k), y'(x_k), \ldots, y^{(n-1)}(x_k)\right) = 0$$
$$j = 1,2,\ldots,n, \qquad (2)$$

va quyidagicha masalani qo'yamiz:

$[a,b]$ oraliqda (1) tenglamaning (2) shartlarni qanoatlantiradigan yechimi topilsin.

Bu *k nuqtali masala* deyiladi. Agar $k = 1$, $x_1 = a$ bo'lganda, Koshi masalasi kelib chiqadi. Agar $k = 2$, $x_1 = a$, $x_2 = b$ bo'lsa, bunday masala chegaraviy masala deyiladi. Va nihoyat qaralyatgan k ta nuqtalardan m tasi ($2 \le m \le k$) turli bo'lsa, u holda (1), (2) *m nuqtali* (yoki *ko'p nuqtali*) *masala* deyiladi.

Chegaraviy yoki ko'p nuqtali (1), (2) masalaning yechimi mavjudligi hamda yechimning yagonaligi isbotlangan, deb faraz qilamiz.

Agar differensial tenglama va chegaraviy shartlar chiziqli bo'lsa, (1), (2) masala chiziqli chegaraviy ($k = 2$) yoki ko'p nuqtali chiziqli ($k > 2$) masala deyiladi.

Differensial tenglama hamda chegaraviy shartlarning kamida bittasi nochiziqli bo'lsa, (1), (2) masala nochiziqli chegaraviy ($k=2$) yoki ko'p nuqtali nochiziqli ($k>2$) masala deyiladi.

Chiziqli chegaraviy yoki ko'p nuqtali masalada differensial tenglama va chegaraviy shartlar birjinsli bo'lsa, u holda (1), (2) birjinsli chegaraviy yoki ko'p nuqtali masala deyiladi. (1) yoki (2) ning birontasi birjinsli bo'lmasa, (1), (2) masala birjinsli bo'lmagan masala deyiladi.

Bir jinsli masala trivial ($y(x)\equiv 0$) yechimga ega. Lekin uning trivial bo'lmagan yechimlarini topish ham ko'p hollarda katta ahamiyatga ega. Buning uchun (1)ga yoki (2)ga biron parametr kiritib, shu parametrga bog'liq bo'lgan notrivial yechim topiladi.

Parametrning bu qiymatlari masalaning xos sonlaridir. Ularga mos keladigan yechimlar masalaning xos funksiyalari deyiladi.

8.2-§. Ikkinchi tartibli chiziqli chegaraviy masalani Koshi masalasiga keltirish

Faraz qilaylik, $[a,b]$ oraliqda

$$y''(x)+p(x)y'(x)+q(x)y(x)=f(x) \tag{1}$$

differensial tenglama berilgan bo'lib, uning

$$\begin{aligned}&\alpha_0 y(a)+\alpha_1 y'(a)=A, \quad |\alpha_0|+|\alpha_1|\neq 0,\\ &\beta_0 y(b)+\beta_1 y'(b)=B, \quad |\beta_0|+|\beta_1|\neq 0.\end{aligned} \tag{2}$$

chegaraviy shartlarni qanoatlantiradigan yechimini topish masalasini ko'ramiz.

Yechimni

$$y(x)=cu(x)+z(x) \tag{3}$$

ko'rinishda qidiramiz, bu yerda c – konstanta, $u(x)$ esa (1) ga mos keluvchi

$$u''(x)+p(x)u'(x)+q(x)u(x)=0 \tag{4}$$

bir jinsli tenglamaning noldan farqli yechimi, $z(x)$

$$z''(x)+p(x)z'(x)+q(x)z(x)=f(x) \tag{5}$$

tenglamaning qandaydir yechimi. (3) bilan aniqlangan $y(x) = cu(x) + z(x)$ yechim ixtiyoriy c da (1) ni qanoatlantirishiga ishonch hosil qilish qiyin emas. Ixtiyoriy c uchun (2) ning birinchisi o'rinli bo'lishini talab etamiz, u holda

$$c\alpha_0 u(a) + \alpha_0 z(a) + c\alpha_1 u'(a) + \alpha_1 z'(a) = A$$

bo'lib, undan

$$c[\alpha_0 u(a) + \alpha_1 u'(a)] + \alpha_0 z(a) + \alpha_1 z'(a) = A \qquad (6)$$

hosil bo'ladi. Bu tenglik ixtiyoriy c larda o'rinli bo'lishi uchun quyidagi tengliklar bajarilishi kerak:

$$\begin{aligned} &\alpha_0 u(a) + \alpha_1 u'(a) = 0, \\ &\alpha_0 z(a) + \alpha_1 z'(a) = A \end{aligned} \qquad (7)$$

Agar ixtiyoriy $c \neq 0$ uchun

$$u(a) = \alpha_1 c, \quad u'(a) = -\alpha_0 c \qquad (8)$$

deb olsak, unda (7) ning birinchisi o'rinli bo'ladi, uning ikkinchisining o'rinli bo'lishligini ta'minlash uchun $\alpha_0 \neq 0$ bo'lganda

$$z(a) = \frac{A}{\alpha_0}, \quad z'(a) = 0 \qquad (9)$$

va $\alpha_1 \neq 0$ bo'lganda

$$z(a) = 0, \quad z'(a) = \frac{A}{\alpha_1} \qquad (10)$$

deb olish mumkin.

Shunday qilib, $u(x)$ (4) birjinsli tenglamaning (8) shartlarni qanoatlantiradigan Koshi masalasining yechimi bo'lib, $z(x)$ esa (9) yoki (10) shartlarni qanoatlantiradigan (5) tenglama uchun Koshi masalasining yechimidir. Shu bilan birga $y(x) = cu(x) + z(x)$ funksiya (2) chegaraviy shartni ixtiyoriy c uchun $x = a$ nuqtada qanoatlantiradi. Ikkinchi chegaraviy shartni $y(x) = cu(x) + z(x)$ funksiya qanoatlantirsin desak, undan c ning qiymati kelib chiqadi, ya'ni

$$[\beta_0 u(b) + \beta_1 u'(b)]c + \beta_0 z(b) + \beta_1 z'(b) = B$$

bundan

$$c = \frac{B - \beta_0 z(b) - \beta_1 z'(b)}{\beta_0 u(b) + \beta_1 u'(b)}$$

hosil bo'ladi. Bu yerda $\beta_0 u(b) + \beta_1 u'(b) \neq 0$ shart bajariladi, deb faraz qilinadi.

Shunday qilib (1), (2) chegaraviy masala (4), (8) va (5), (9) (yoki (10)) Koshi masalasini yechishga keltirildi.

Shuni ta'kidlash lozimki, agar $\beta_0 u(b)+\beta_1 u'(b)\neq 0$ bo'lsa, (1), (2) chegaraviy masala yagona yechimga ega bo'ladi, aks holda bu masala yo umuman yechimga ega emas, yoki cheksiz ko'p yechimga ega bo'ladi.

8.3-§. Chekli-ayirmali metod bilan ikkinchi tartibli chegaraviy masalani yechish

Faraz qilaylik, $[a,b]$ da quyidagi

$$L(y) \equiv y''(x)+ p(x)\,y'(x)+ q(x)\,y(x)= f(x) \qquad (1)$$

$$\alpha_0\, y(a)+\alpha_1\, y'(a)= A, \qquad |\alpha_0|+|\alpha_1|\neq 0, \qquad (2)$$

$$\beta_0\, y(b)+\beta_1\, y'(b)= B, \qquad |\beta_0|+|\beta_1|\neq 0 \qquad (3)$$

chegaraviy masala berilgan bo'lib, u yagona yechimga ega bo'lsin.

$[a,b]$ oraliqni $h=(b-a)/N$ qadam bilan teng N bo'lakka bo'lib, $\{x_i\}_{i=0}^{n}$ to'r hosil qilamiz:

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b. \qquad (4)$$

Endi (1) ni $x_1, x_2, \ldots, x_{N-1}$ nuqtalarda, ya'ni $\{x_i\}_{i=0}^{N}$ to'rning ichki nuqtalarida, (2) va (3) ni mos ravishda x_0, x_N nuqtalarda qaraymiz. (1) da $x = x_i$ $i=1,2,\ldots,N-1$ desak,

$$y''(x_i)+ p(x_i)\,y'(x_i)+ q(x_i)\,y(x_i)= f(x_i),\ \ i=1,2,\ldots,N-1, \qquad (5)$$

hosil bo'ladi. Bu yerda $y'(x_i)$, $y''(x_i)$ larni $y(x)$ funksiya qiymatlari orqali approksimatsiya qilamiz. Buning uchun x_i nuqta atrofida $y(x)$ to'rtinchi tartibli hosilaga ega, deb hisoblaymiz va quyidagi yoyilmalarni hosil qilamiz:

$$y(x_{i+1})=y(x_i+h)=y(x_i)+\frac{h}{1!}y'(x_i)+\frac{h^2}{2!}y''(x_i)+\frac{h^3}{3!}y'''(x_i)+\frac{h^4}{4!}y^{(IV)}(x_i+\theta h), \qquad (6)$$

$$y(x_{i-1}) = y(x_i - h) = y(x_i) - \frac{h}{1!}y'(x_i) + \frac{h^2}{2!}y''(x_i) - \frac{h^3}{3!}y'''(x_i) + \frac{h^4}{4!}y^{(IV)}(x_i - \theta_1 h), \tag{7}$$

$$0 < \theta < 1, \quad 0 < \theta_1 < 1.$$

Bulardan quyidagilarga ega bo'lamiz:

$$\frac{y(x_{i+1}) - y(x_i)}{h} = y'(x_i) + o(h), \tag{8}$$

$$\frac{y(x_i) - y(x_{i-1})}{h} = y'(x_i) + o(h), \tag{9}$$

$$\frac{y(x_{i+1}) - y(x_{i-1})}{2h} = y'(x_i) + o(h^2). \tag{10}$$

Bularning chap tomoni mos ravishda *o'ng hosila*, *chap hosila* va *markaziy hosila* deb ataladi. Shunga o'xshash $y''(x_i)$ uchun

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = y''(x_i) + o(h^2) \tag{11}$$

formulani hosil qilish mumkin.

Endi (5)dan (10), (11) larga asosan

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} + p(x_i)\frac{y(x_{i+1}) - y(x_{i-1})}{2h} + q(x_i)y(x_i) = f(x_i) + o(h^2) \tag{12}$$

ni hosil qilamiz, bundan esa

$$\left[1 - \frac{h}{2}p(x_i)\right]y(x_{i-1}) - \left[2 - h^2 q(x_i)\right]y(x_i) + \left[1 + \frac{h}{2}p(x_i)\right]y(x_{i+1}) = \\ = h^2 f(x_i) + o(h^4), \qquad i = 1, 2, \ldots, N-1 \tag{13}$$

ko'rinishga ega bo'lgan (5) ning to'rning ichki nuqtalaridagi approksimatsiyasi hosil bo'ladi.

Shuningdek, (2), (3) chegaraviy shartlarning (8), (9) formulalarni ishlatib approksimatsiyasini topamiz:

$$(\alpha_0 h - \alpha_1)y(x_0) + \alpha_1 y(x_1) = hA + o(h^2), \tag{14}$$

$$-\beta_1 y(x_{n-1}) - (\beta_1 + h\beta_0)y(x_n) = hB + o(h^2). \tag{15}$$

Endi (13), (14), (15) lardagi $o(h^4)$ va $o(h^2)$ larni tashlab yuborib hamda quyidagi belgilashlarni kiritib

$$A_i = 1 - \frac{h}{2} p(x_i), \quad C_i = 2 - h^2 q(x_i), \quad B_i = 1 + \frac{h}{2} p(x_i), \quad y(x_i) = y_i,$$

$$\chi_1 = \frac{\alpha_1}{\alpha_1 - h\alpha_0}, \quad \mu_1 = \frac{hA}{h\alpha_0 - \alpha_1}, \quad \chi_2 = \frac{\beta_1}{\beta_1 + h\beta_0}, \quad \mu_2 = \frac{hB}{\beta_1 + h\beta_0}$$

quyidagi tenglamalar sistemasini

$$\left.\begin{array}{l} y_0 = \chi_1 y_1 + \mu_1, \\ A_i y_{i-1} - C_i y_i + B_i y_{i+1} = h^2 f_i, \quad i = 1,2,\ldots,N-1, \\ y_N = \chi_2 y_{N-1} + \mu_2 \end{array}\right\} \qquad (16)$$

hosil qilamiz.

Bu tenglamalar sistemasining matritsasi uch diagonalli. Bunday sistemalarni yechish uchun odatda *haydash usuli* qo'llanadi. (16) ning yechimini quyidagi

$$y_{i+1} = a_{i+1} y_i + b_{i+1}, \quad i = 0,1,\ldots,N-1 \qquad (17)$$

ko'rinishda izlaymiz, bu yerda a_{i+1}, b_{i+1} lar noma'lumlar. (17) dan quyidagilarga ega bo'lamiz:

$$y_i = a_i y_{i-1} + b_i ,$$

$$y_{i+1} = a_{i+1} y_i + b_{i+1} = a_i a_{i+1} y_{i-1} + a_{i+1} b_i + b_{i+1}.$$

Bu ifodalarni (16) ga qo'ysak,

$$\left[A_i - a_i\left(C_i - B_i a_{i+1}\right)\right] y_{i-1} + \left[b_i\left(B_i a_{i+1} - C_i\right) + B_i b_{i+1} - h^2 f_i\right] = 0$$

hosil bo'ladi. Agar

$$a_i = \frac{A_i}{C_i - B_i a_{i+1}}, \quad b_i = \frac{B_i b_{i+1} - h^2 f_i}{C_i - B_i a_{i+1}}, \quad i = 1,2,\ldots,N-1$$

bo'lsa, yuqoridagi tenglik o'rinli bo'ladi. a_N va b_N larni esa (16) ning oxirgi tenglamasidan va (17) da $i = N-1$ bo'lganda topamiz:

$$a_N = \chi_2, \quad b_N = \mu_2.$$

(16) ning birinchisi va (17) da $i = 0$ desak

$$y_0 = \frac{\chi_1 b_1 + \mu_1}{1 - \chi_1 a_1}$$

hosil bo'ladi.

Shunday qilib, (16) tenglamalar sistemasini haydash usuli bilan yechish algoritmini hosil qildik:

$$\left.\begin{aligned} &a_i=\frac{A_i}{C_i-B_i a_{i+1}}, \quad b_i=\frac{B_i b_{i+1}-h^2 f_i}{C_i-B_i a_{i+1}}, \\ &i=1,2,\ldots,N-1; \\ &a_N=\chi_2, \qquad b_N=\mu_2, \\ &y_{i+1}=a_{i+1}y_i+b_{i+1}, \quad i=0,1,\ldots,N-1 \\ &y_0=(\mu_1+\chi_1 a_1)/(1-\chi_1 a_1) \end{aligned}\right\} \qquad (18)$$

Bu algoritm bo'yicha y_i lar chegaraning chap nuqtasidan boshlab ketma-ket topiladi, shuning uchun (18) formulalar *chapdan haydash formulalari* deyiladi. Xuddi shuningdek, o'ngdan haydash formulalarini chiqarish mumkin:

$$\left.\begin{aligned} &\xi_{i+1}=\frac{B_i}{C_i-\xi_i A_i}, \quad \eta_{i+1}=\frac{\eta_i A_i-h^2 f_i}{C_i-\eta_i A_i}, \\ &\xi_1=\chi_1, \quad \eta_1=\mu_1, \\ &y_i=\xi_{i+1}y_{i+1}+\eta_{i+1}, \quad i=0,1,\ldots,N-1 \\ &y_n=(\mu_2+\chi_2\eta_N)/(1-\chi_2\xi_N) \end{aligned}\right\} \qquad (19)$$

Agar a_i koeffitsiyentlar moduli bo'yicha birdan kichik bo'lsa, u holda (18) haydovchi formulalar *turg'un* deyiladi.

Quyidagi teorema o'rinlidir.

Teorema. Agar

$$A_i>0, \quad B_i>0, \quad C_i\geq A_i+B_i, \quad i=1,2,\ldots,N-1;$$

$$0\leq \chi_1, \ \chi_2<1$$

shartlar bajarilsa, u holda haydashni bajarish mumkin va u turg'un bo'ladi.

Isboti. Haqiqatan ham, $0\leq a_N=\chi_2<1$ ekanligi teorema shartidan kelib chiqadi. Faraz qilaylik, $0\leq a_{i+1}<1$ bo'lsin, u holda

$$0<a_i=\frac{A_i}{C_i-B_i a_{i+1}}=\frac{A_i}{(C_i-B_i-A_i)+A_i+(1-a_{i+1})B_i}<1$$

bo'ladi, chunki maxraj suratdan katta. Demak, $i=1,2,\ldots,N-1$ uchun $0<a_i<1$. Endi $0\leq\chi_i<1$ shart $0<a_i<1$ bilan birgalikda y_0 ni aniqlaydigan formulada maxraj noldan farqligini ta'minlaydi.

8.4-§. Taqribiy analitik metodlar

Ba'zi bir tatbiqiy masalalarni yechishda to'r usulini emas, balki yechimni, taqribiy bo'lsada, analitik ko'rinishda topish talab etiladi. Bunday hollarda yechimning aniqligi darajasi yuqori bo'lishligi talab qilinmaydi, ammo aniq yechim o'rniga shunday funksiyani qurish talab etiladiki, u chegaraviy shartlarni qanoatlantirishi kerak hamda differensial tenglamaga bog'liq bo'lgan qandaydir munosabatlarning bajarilishini talab etadi. Yuqoridagi talablarga javob beruvchi funksiya taqribiy ravishda differensial tenglamani ham qanoatlantiradi. Bu munosabatlarni hosil qilishda qo'yiladigan talablarga qarab turli metodlar hosil bo'ladi.

8.4.1. Kollokatsiya metodi

Faraz qilaylik, quyidagi

$$L(y) \equiv y''(x) + p(x)\,y'(x) + q(x)\,y(x) = f(x) \qquad (1)$$

$$l_a(y) \equiv \alpha_0\, y(a) + \alpha_1\, y'(a) = A, \quad |\alpha_0| + |\alpha_1| \neq 0, \qquad (2)$$

$$l_b(y) \equiv \beta_0\, y(b) + \beta_1\, y'(b) = B, \quad |\beta_0| + |\beta_1| \neq 0. \qquad (3)$$

chegaraviy masala berilgan bo'lsin. Bu masalaning taqribiy yechimini

$$y_n(x) = \varphi_0(x) + \sum_{k=1}^{n} c_k \varphi_k(x) \qquad (4)$$

ko'rinishda izlaymiz, bu yerda $\varphi_k(x)$ $(k = 0,1,...,n)$ – ma'lum funksiyalar bo'lib, ular hozircha

$$l_a(\varphi_0) = A,\; l_b(\varphi_0) = B,\; l_a(\varphi_k) = l_b(\varphi_k) = 0,\; k = 1,2,...,n$$

shartlarni qanoatlantirishi talab qilinadi. U holda ixtiyoriy c_k $(k = 1,2,...,n)$ lar uchun $y_n(x)$ (2), (3) chegaraviy shartlarni qanoatlantiradi, ammo u differensial tenglamani qanoatlantirmasligi mumkin, ya'ni

$$L(y_n) \neq f(x).$$

$y_n(x)$ (1)ning aniq yechimi $y(x)$ bilan aynan bir xil bo'lgan hol bundan istisno. $L(y_n) = f(x)$ tenglikni $[a,b]$ ga tegishli turli

$x_1, x_2, \dots, x_n$ nuqtalarda bajarilishini talab qilsak, noma'lum c_k $(k = 1,2,\dots,n)$ parametrlarni topish uchun

$$\sum_{k=1}^{n} c_k L(\varphi_k(x_i)) = f(x_i) - L(\varphi_0(x_i)), \quad i = 1,2,\dots,n \qquad (5)$$

ko'rinishga ega chiziqli algebraik tenglamalar sistemasini hosil qilamiz. Bu tenglamalar sistemasi yagona yechimga ega bo'lishi uchun $\varphi_k(x)$ $(k = 1,2,\dots,n)$ lar quyidagi shartlarni qanoatlantirishi kerak:

1) bu funksiyalar chiziqli bog'liqsiz bo'lishi kerak;

2) $p(x), q(x), f(x)$ funksiyalar $[a,b]$ da uzluksiz bo'lsa, $\varphi_k(x) \in C_2[a,b]$ bo'lishi kerak;

3) $\varphi_k(x)$ $(k = 1,2,\dots,n)$ funksiyalar yordamida hosil qilingan sistema $L(\varphi_k(x))$ $(k = 1,2,\dots,n)$ ixtiyoriy va turli x_i $(i = 1,2,\dots,n)$ nuqtalar uchun Chebishev sistemasini tashkil qilishi kerak.

Ta'rif. Agar $[a,b]$ ga tegishli va turli x_i $(i = 1,2,\dots,n)$ nuqtalar uchun

$$\begin{vmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \dots & \varphi_n(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) & \dots & \varphi_n(x_2) \\ \dots & \dots & \dots & \dots \\ \varphi_1(x_n) & \varphi_2(x_n) & \dots & \varphi_n(x_n) \end{vmatrix} \neq 0$$

bo'lsa, $\varphi_k(x)$ $(k = 1,2,\dots,n)$ funksiyalar sistemasi $[a,b]$ da Chebishev sistemasini tashkil etadi deyiladi.

Endi (5) ni yechib, noma'lum parametr c_k $(k = 1,2,\dots,n)$ larni topamiz va (1)–(3) chegaraviy masala yechimini

$$y_n(x) = \varphi_0(x) + \sum_{k=1}^{n} c_k \varphi_k(x)$$

ko'rinishda topgan bo'lamiz.

Shuni eslatib o'tamizki, bu metoddagi

$$L(y_n(x))\Big|_{x=x_i} = L(\varphi_0(x))\Big|_{x=x_i} + \sum_{k=1}^{n} c_k L(\varphi_k(x))\Bigg|_{x=x_i}, \quad i = 1,2,\dots,n$$

shartlarning bajarilishi $L(y_n(x))$ ning $L(\varphi_k(x))$ funksiyalar sistemasi bo'yicha x_i $(i=1,2,...,n)$ nuqtalarda interpolyatsiyalanishini anglatadi. Bunday interpolyatsiyalash mumkin va yagona bo'lishligi uchun $[a,b]$ da $L(\varphi_k(x))$ funksiyalar sistemasi Chebishev sistemasidan iborat bo'lishligi kerak.

8.4.2. Kichik kvadratlar metodi

Faraz qilaylik, H – haqiqiy Gilbert fazosi bo'lsin. H da zich $D(A)$ to'plamda aniqlangan, qiymatlari H ga tegishli chiziqli operatorni A deb belgilaylik va

$$Ay=f \tag{1}$$

tenglamani ko'raylik. Bu yerda $f\in H$ va yechim $y\in D(A)$. (1) tenglama yechimga ega bo'lsin deylik.

(1) tenglamaga quyidagi

$$J(y)=\|Ay-f\|^2 \tag{2}$$

funksionalni mos qo'yamiz va (1) ni yechish masalasini shu funksionalni $D(A)$ da minimumga erishtiruvchi elementni aniqlash masalasiga almashtiramiz, ya'ni

$$\min_{y\in D(A)} J(y)=J(y^*)=0.$$

Bu tenglamadan ayonki, y^* $Ay=f$ tenglamaning yechimi bo'ladi. $Ay=f$ tenglamani yechish o'rniga (2) funksionalning minimumini topishga asoslanib, $Ay=f$ tenglamani yechish metodi *kichik kvadratlar metodi* deyiladi.

$J(y)$ funksionalni quyidagicha minimumga erishtiramiz. $D(A)$ ga tegishli chiziqli erkli $\{\varphi_k(x)\}$ funksiyalar sistemasini tanlaymiz va (1) tenglamaning yechimiga n-yaqinlashishni

$$y_n(x)=\varphi_0(x)+\sum_{k=1}^{n}c_k\varphi_k(x) \tag{3}$$

ko'rinishda izlaymiz. Noma'lum c_k $(k=1,2,...,n)$ koeffitsiyentlar

$$J(y_n)=\|Ay_n-f\|^2$$

funksional minimumga erishsin deb topiladi.

Biz quyidagi

$$L(y)\equiv y''(x)+p(x)y'(x)+q(x)y(x)=f(x) \tag{4}$$

$$\begin{aligned} l_a(y)&\equiv\alpha_0\, y(a)+\alpha_1\, y'(a)=A, \quad |\alpha_0|+|\alpha_1|\neq 0,\\ l_b(y)&\equiv\beta_0\, y(b)+\beta_1\, y'(b)=B, \quad |\beta_0|+|\beta_1|\neq 0 \end{aligned} \tag{5}$$

chiziqli chegaraviy masalani kichik kvadratlar metodi bilan yechish sxemasini keltiramiz.

Quyidagi shartlarni qanoatlantiruvchi $\varphi_k(x)$ $(k=1,2,...,n)$ funksiyalar sistemasini qaraymiz:

1. $\varphi_k(x)\in C_2[a,b]$.

2. $l_a(\varphi_0)=A,\ l_b(\varphi_0)=B,\ l_a(\varphi_k)=l_b(\varphi_k)=0,\ k=1,2,...,n$.

3. Ixtiyoriy chekli n uchun $\{\varphi_k(x)\}_{k=1}^{n}$ funksiyalar sistemasi chiziqli erkli.

4. $\varphi_k(x)$ $(k=1,2,...,n)$ $C_2[a,b]$ da to‘liq bo‘lsin.

(4), (5) chegaraviy masalaning taqribiy yechimini (3) ko‘rinishda izlaymiz. (3) funksiya (5) chegaraviy shartlarni ixtiyoriy n va $c_1,c_2,...,c_n$ lar uchun qanoatlantiradi. $c_1,c_2,...,c_n$ larni shunday tanlaylikki,

$$J(y_n)=\|Ay_n-f\|^2$$

minimal qiymatga ega bo‘lsin.

Ma’lumki,

$$\|L(y_n)-f\|^2=\left(L(y_n)-f,L(y_n)-f\right)=$$

$$=\int_a^b\left[L(\varphi_0)+\sum_{k=1}^{n}c_kL(\varphi_k(x))-f(x)\right]^2dx=J(c_1,c_2,...,c_n).$$

Bundan c_i $(i=1,2,...,n)$ lar bo‘yicha birinchi tartibli xususiy hosilalar olib nolga tenglaymiz va hosil bo‘lgan chiziqli algebraik tenglamalarni hosil qilamiz:

$$\frac{\partial J}{\partial c_i}=2\int_a^b\left[L(\varphi_0(x))+\sum_{k=1}^{n}c_kL(\varphi_k(x))-f(x)\right]\cdot L(\varphi_i(x))dx=0,$$

$$i=1,2,\ldots,n.$$

Buni quyidagicha yozamiz:

$$\sum_{k=1}^{n}A_{ik}c_k=-B_i,\quad i=1,2,\ldots,n \tag{6}$$

bu yerda

$$A_{ik}=A_{ki}=\int_a^b L(\varphi_k(x))L(\varphi_i(x))dx,\ B_i=\int_a^b\left[\left(L(\varphi_0(x))-f(x)\right)\cdot L(\varphi_i(x))\right]dx,$$

$$i,k=1,2,\ldots,n.$$

(6) tenglamalar sistemasining matritsasi $\left\{L(\varphi_k(x))\right\}_{k=1}^{n}$ funksiyalar sistemasi uchun tuzilgan Gram matritsasidir.

(6) tenglamalar sistemasining yechimga ega bo‘lishi faqat $\{\varphi_k(x)\}$ funksiyalar sistemasining xususiyatlariga bog‘liq bo‘lmay, qaralayotgan chegaraviy masalaga ham bog‘liq bo‘ladi, xususan,

$$L(y)=0,\ \ y(a)=0,\ \ y(b)=0 \tag{7}$$

bir jinsli chegaraviy masala faqat trivial yechimga ega bo‘lishi kerak.

Quyidagi teorema o‘rinli.

Teorema. Agar (7) chegaraviy masala faqat nol yechimga ega bo‘lsa, u holda (6) chiziqli algebraik tenglamalar sistemasining matritsasi maxsusmas bo‘lib, (6) yagona yechimga ega.

8.4.3. Galerkin metodi

Bu metodda ham kichik kvadratlar metodidagi (4), (5) chegaraviy masalani yechishdagi $\{\varphi_k(x)\}$ funksiyalar sistemasiga bog‘liq shartlar o‘rinli deb hisoblanadi. (3) ko‘rinishdagi taqribiy yechimning $c_1,c_2,\ldots,c_n$ parametrlari bu metodda quyidagi shartlardan topiladi:

$$\left(L(y_n(x))-f(x)\ ,\ \varphi_i(x)\right)=0,\quad i=1,2,\ldots,n.$$

Buni quyidagicha yozamiz:

$$\sum_{k=1}^{n}c_k\left(L(\varphi_k(x))\ ,\ \varphi_i(x)\right)=\left(f(x)-L(\varphi_0(x))\ ,\ \varphi_i(x)\right),\ i=1,2,\ldots,n$$

Bu chiziqli algebraik tenglamalar sistemasini yechib noma'lum parametrlarni aniqlaymiz va (4), (5) chegaraviy masalaning taqribiy yechimini

$$y_n(x)=\sum_{k=1}^{n} c_k\varphi_k(x)$$

ko'rinishda aniqlagan bo'lamiz.

Bobga tegishli tayanch so'zlar: k nuqtali masala, chegaraviy masala, ayirmali metod, haydash usuli, turg'unlik, taqribiy analitik metodlar.

Savollar va topshiriqlar

1. Chegaraviy masala tushunchasi va ularning turlari.
2. Ikkinchi tartibli chiziqli chegaraviy masalani Koshi masalasiga keltirish.
3. Birinchi va ikkinchi tartibli hosilalarni to'r sohadagi approksimatsiyalari. Approksimatsiya tartibi.
4. Chekli – ayirmali metod bilan ikkinchi tartibli chegaraviy masalani yechish.
5. Haydash usuli algoritmini hosil qiling.
6. O'ng va chap haydash usullarini tushuntiring.
7. Haydash usulining turg'unlik tushunchasi.
8. Kollokatsiya metodi.
9. Kichik kvadratlar metodi.
10. Galerkin metodi.

Misol 1.

$y''-2xy'-2y=-4x;$

$y(0)-y'(0)=0;$

$y\left(\frac{1}{2}\right)=1,7840.$

Aniq yechim: $y(x)=x+e^{x^2}$.

Yechish. $h=0{,}1$ bo'lsin. Bu yerda $p(x)=-2x$, $q(x)=2$, $f(x)=-4x$, $\alpha_0=1$, $\alpha_1=-1$, $A=0$, $\beta_1=0$, $B=1{,}7840$.

Tenglamani x_i $i=1,2,3,4$ nuqtalarda, ya'ni to'rning ichki nuqtalarida, chegaraviy shartlarning esa chegaraviy nuqtalarda approksimatsiyasini yozamiz va quyidagi

$$\begin{aligned} y_0 &= \chi_1 y_1+\mu_1, \\ A_i y_{i-1}-C_i y_i+B y_{i+1} &= h^2 f_i, \quad i=1,2,3,4, \\ y_5 &= \chi_2 y_4+\mu_2 \end{aligned} \tag{1}$$

sistemani hosil qilamiz, bu yerda

$$\begin{aligned} A_i &= 1-\frac{h}{2}p(x_i)=1-\frac{0{,}1}{2}(-2x_i)=1+0{,}1x_i, \\ B_i &= 1+\frac{h}{2}p(x_i)=1+\frac{0{,}1}{2}(-2x_i)=1-0{,}1x_i, \\ C_i &= 2-h^2q(x_i)=2-0{,}01\cdot(-2)=2{,}02, \end{aligned} \tag{2}$$

$$\chi_1=\frac{\alpha_1}{\alpha_1-h\alpha_0}=\frac{-1}{-1-0{,}1\cdot 1}=\frac{1}{1{,}1}, \quad \mu_1=\frac{hA}{h\alpha_0-\alpha_1}=\frac{h\cdot 0}{1+1}=0,$$

$$\chi_2=\frac{\beta_1}{\beta_1+h\beta_0}=\frac{0}{0+h\cdot 1}=0, \quad \mu_2=\frac{hB}{h\beta_0+\beta_1}=\frac{0{,}1\cdot 1{,}7840}{0+0{,}1\cdot 1}=1{,}7840,$$

$$h^2 f_i=0{,}01\cdot(-4x_i)=-0{,}04x_i, \quad i=1,2,3,4.$$

(1)ni (2)ni e'tiborga olib, haydash usulini qo'llab yechamiz. Uning yechimini $y_{i+1}=a_{i+1}y_i+b_{i+1}$, $i=0,1,2,3,4$ ko'rinishda izlaymiz. Haydash koeffitsiyentlari esa

$$a_i=\frac{A_i}{C_i-B_i a_{i+1}}, \quad b_i=\frac{B_i b_{i+1}-h^2 f_i}{C_i-B_i a_{i+1}}, \quad i=1,2,3,4 \tag{3}$$

formulalar bilan aniqlanadi. a_5 va b_5 lar esa $y_5=\chi_2 y_4+\mu_2$ va $y_5=a_5y_4+b_5$ lardan topiladi. Ular $a_5=\chi_2=0$, $b_5=\mu_2=1{,}7840$.

Endi (3) formula yordamida ketma-ket $a_4, b_4, a_3, b_3, a_2, b_2, a_1, b_1$ larni hisoblaymiz:

$$a_4=\frac{A_4}{C_4-B_4a_5}=\frac{1+0{,}1\cdot 0{,}4}{2{,}02-(1-0{,}04)\cdot 0}=\frac{1{,}04}{2{,}02}=0{,}51485,$$

$$a_3=\frac{A_3}{C_3-B_3a_4}=\frac{1+0{,}1\cdot 0{,}3}{2{,}02-(1-0{,}1\cdot 0{,}3)\cdot 0{,}51485}=\frac{1{,}03}{1{,}52070}=0{,}67732,$$

$$a_2=\frac{A_2}{C_2-B_2a_3}=\frac{1+0{,}1\cdot 0{,}2}{2{,}02-(1-0{,}1\cdot 0{,}2)\cdot 0{,}67732}=\frac{1{,}02}{1{,}357223}=0{,}75147,$$

$$a_1=\frac{A_1}{C_1-B_1a_2}=\frac{1+0{,}1\cdot 0{,}1}{2{,}02-(1-0{,}1\cdot 0{,}1)\cdot 0{,}75147}=\frac{1{,}01}{1{,}27904}=0{,}78964,$$

$$b_4=\frac{B_4\cdot b_5-h^2f_4}{C_4-B_4a_5}=\frac{(1-0{,}1\cdot 0{,}4)\cdot 1{,}7840+0{,}01\cdot 4\cdot 0{,}4}{2{,}02}=0{,}85576,$$

$$b_3=\frac{B_3b_4-h^2f_3}{C_3-B_3a_4}=\frac{(1-0{,}1\cdot 0{,}3)\cdot 0{,}85576+0{,}01\cdot 4\cdot 0{,}2}{1{,}52070}=0{,}55369,$$

$$b_2=\frac{B_2\cdot b_3-h^2f_2}{C_2-B_2a_3}=\frac{(1-0{,}1\cdot 0{,}2)\cdot 0{,}55369+0{,}01\cdot 4\cdot 0{,}2}{1{,}35723}=0{,}40422,$$

$$b_1=\frac{B_1b_2-h^2f_1}{C_1-B_1a_2}=\frac{(1-0{,}1\cdot 0{,}1)\cdot 0{,}40422+0{,}01\cdot 4\cdot 0{,}1}{1{,}27904}=0{,}31600.$$

(1)ning birinchi ifodasidan va $y_1=a_1y_0+b_1$ dan $y_0=1{,}10199$ ekanligi kelib chiqadi. Endi y_1, y_2, y_3, y_4 larni va (1) ning oxirgisi ifodasidan y_5 ni aniqlaymiz. Ularni keltiramiz:

$y_1=1{,}18617;\ y_2=1{,}27548;\ y_3=1{,}33283;\ y_4=1{,}54196;\ y_5=1{,}7840.$

Misol 2.

$$y''(x)+2xy'(x)-2y(x)=2,$$
$$y'(0)=-2,$$
$$y(1)+y'(1)=0$$

chegaraviy masalani $n=2$ da kollokatsiya metodi bilan yeching.

Yechish. $\varphi_0(x)$ ni $\varphi_0(x)=b+cx$ ko'rinishda izlaymiz:

$$\varphi_0'(x)=c,\ \varphi_0'(0)=-2\Rightarrow c=-2.$$

$$\varphi_0'(1)+\varphi_0'(1)=0\Rightarrow b+c+c=0\Rightarrow b=4.$$

Demak, $\varphi_0(x)=4-2x$. $\varphi_k(x)\ (k=1,2,\ldots,)$ funksiyalar

$$\left.\begin{array}{l}\varphi_k'(0)=0\\ \varphi_k(1)+\varphi_k'(1)=0\end{array}\right\}$$

shartlarni qanoatlantirishi kerak.

$\varphi_k(x)=b_k+x^{k+1}$ desak, b_k koeffitsiyentni ikkinchi tenglamadan topamiz:

$$1+b_k+(k+1)=0\Rightarrow b_k=-(k+2),\ \ \varphi_k(x)=x^{k+1}-(k+2).$$

Yechimni $y_2(x)=4-2x+c_1(x^2-3)+c_2(x^3-4)$ ko'rinishda izlaymiz. $L(\varphi_0(x))$, $L(\varphi_1(x))$, $L(\varphi_2(x))$ larni hisoblab, $R(x,c_1,c_2)$ ni topamiz:

$\varphi_0(x)=4-2x,\ \ \varphi_0'(x)=-2,\ \ \varphi_0''(x)=0,$

$L(\varphi_0(x))=\varphi_0''(x)+2x\varphi_0'(x)-2\varphi_0(x)=-4x-8+4x=-8,$

$\varphi_1(x)=x^2-3,\ \ \varphi_1'(x)=2x,\ \ \varphi_1''(x)=2,$

$L(\varphi_1(x))=\varphi_1''(x)+2x\varphi_1'(x)-2\varphi_1(x)=2+2x\cdot 2x-2(x^2-3)=2x^2+8,$

$\varphi_2(x)=x^3-4,\ \ \varphi_2'(x)=3x^2,\ \ \varphi_2''(x)=6x,$

$L(\varphi_2(x))=\varphi_2''(x)+2x\varphi_2'(x)-2\varphi_2(x)=6x+2x\cdot 3x^2-2x^3+8=4x^3+6x+8,$

$R(x,c_1,c_2)=L(\varphi_0(x))-f(x)+c_1L(\varphi_1(x))+c_2L(\varphi_2(x)).$

Kollokatsiya nuqtalarini $x_1=\frac{1}{4}$, $x_2=\frac{3}{4}$ deb olaylik, u holda quyidagilar hosil bo'ladi:

$L(\varphi_0(x_1))=-8,\ \ L(\varphi_0(x_2))=-8,\ \ f(x_1)=f(x_2)=2,$

$L(\varphi_1(x_1))=2\cdot\frac{1}{16}+8=8\frac{1}{8},$

$L(\varphi_1(x_2))=2\cdot\frac{9}{16}+8=9\frac{1}{8},$

$L(\varphi_2(x_1))=4\cdot\frac{1}{64}+6\cdot\frac{1}{4}+8=14\frac{3}{16}.$

Natijada

$$\left.\begin{aligned}\frac{65}{8}c_1+\frac{153}{16}c_2=10\\ \frac{73}{8}c_1+\frac{227}{16}c_2=10\end{aligned}\right\}$$

tenglamalar sistemasiga ega bo'lamiz, bundan $c_1=1{,}6510$, $c_2=-0{,}3570$ ekanligini aniqlaymiz va taqribiy analitik yechim

$$y_2(x)=4-2x+1{,}6510(x^2-3)-0{,}3570(x^3-4)$$

ko'rinishda topilgan bo'ladi.

Misol 3.

$$L(y) \equiv y''(x) + 2xy'(x) - 2y(x) = 2,$$
$$l_0(y) \equiv y'(0) = -2,$$
$$l_1(y) \equiv y(1) + y'(1) = 0$$

chegaraviy masalani $n = 2$ da kichik kvadratlar usuli bilan yeching.

Yechish. Kollokatsiya metodi bilan bu masalani yechganimizda $\varphi_0(x), \varphi_1(x), \varphi_2(x)$ lar aniqlangan edi, shuningdek, $L(\varphi_0(x))$, $L(\varphi_1(x))$, $L(\varphi_2(x))$ lar ham hisoblangandi.

Ulardan foydalanib, (6) sistemaning koeffitsiyentlarini hisoblaymiz:

$$A_{11} = \int_0^1 L^2(\varphi_1(x))dx = \int_0^1 (2x^2 + 8)^2 dx = \int_0^1 (4x^4 + 32x^2 + 64)dx =$$
$$= \frac{4}{5} + \frac{32}{3} + 64 = \frac{1132}{15},$$

$$A_{12} = A_{21} = \int_0^1 (2x^2 + 8)(4x^3 + 6x + 8)dx =$$
$$= \int_0^1 (8x^5 + 16x^3 + 16x^2 + 32x^3 + 48x + 64)dx =$$
$$= \frac{8}{6} + \frac{48}{4} + \frac{16}{3} + \frac{48}{2} + 64 = \frac{320}{3};$$

$$A_{22} = \int_0^1 (4x^3 + 6x + 8)dx = \int_0^1 (16x^6 + 36x^2 + 64 + 48x^4 + 64x^3 + 96x)dx =$$
$$= \frac{16}{7} + \frac{36}{3} + 64 + \frac{48}{5} + \frac{64}{4} + \frac{96}{2} = \frac{5316}{35};$$

$$B_1 = \int_0^1 (L(\varphi_0(x)) - 2)L(\varphi_1(x))dx = \int_0^1 (-8 - 2)(2x^2 + 8)dx =$$
$$= -10\left(\frac{2}{3} + 8\right) = \frac{-260}{3};$$

$$B_2 = \int_0^1 (L(\varphi_0(x)) - 2)L(\varphi_2(x))dx = \int_0^1 (-10)(4x^3 + 6x + 8)dx =$$
$$= -10\left(\frac{4}{4} + \frac{6}{2} + 8\right) = -120.$$

Endi (6) tenglamalar sistemasini yozamiz:

$$\left.\begin{aligned}\frac{1132}{15}c_1+\frac{320}{3}c_2=\frac{260}{3}\\ \frac{320}{3}c_1+\frac{5316}{35}c_2=120\end{aligned}\right\}\Rightarrow c_1\cong 0,7556,\ \ c_2\cong 0,2779$$

Berilgan chegaraviy masalaning taqribiy analitik echimi

$$y_2(x)=4-2x+0,7556(x^2-3)+0,2779(x^3-4)$$

ko'rinishda bo'ladi.

Misol 4.

$$y''+y+x=0$$
$$y(0)=y(1)=0$$

chegaraviy masalani $n=3$ da Galerkin metodi bilan yeching.

Yechish. $\varphi_k(x)$ funksiyalar sifatida quyidagilarni olamiz:

$$\varphi_0(x)\equiv 0,\ \ \varphi_1(x)=x(1-x),\ \varphi_2(x)=x^2(1-x),\ ...,\ \varphi_n(x)=x^n(1-x).$$

Taqribiy yechimni $n=3$ da

$$y_3(x)=\sum_{k=1}^{3}c_k\varphi_k(x)=\sum_{k=1}^{3}c_kx^k(1-x)$$

ko'rinishda izlaymiz. Nomalum c_k $(k=1,2,3)$ larni quyidagi

$$\sum_{k=1}^{3}c_k(L(\varphi_k(x)),\varphi_i(x))=(f(x),\varphi_i(x)),\ \ i=1,2,3$$

tenglamalar sistemasidan aniqlaymiz. Buning uchun avval $L(\varphi_k(x))$, $(k=1,2,3)$ larni hisoblaymiz, so'ng yuqoridagi tenglamalar sistemasidagi skalyar ko'paytmalarni hisoblaymiz:

$$\varphi_1(x)=x-x^2,\ \ \varphi_1'(x)=1-2x,\ \ \ \varphi_1''(x)=-2,\ \ L(\varphi_1(x))=-2+x-x^2,$$

$$\varphi_2(x)=x^2-x^3,\ \varphi_2'(x)=2x-3x^2,\ \ \varphi_2''(x)=2-6x,\ L(\varphi_2(x))=2-6x+x^2-x^3,$$

$$\varphi_3(x)=x^3-x^4,\ \ \varphi_3'(x)=3x^2-4x^3,\ \ \varphi_3''(x)=6x-12x^2,$$

$$L(\varphi_3(x))=6x-12x^2+x^3-x^4,$$

$$(L(\varphi_1(x)),\varphi_1(x))=\int_0^1(-2+x-x^2)(x-x^2)dx=-\frac{3}{10},$$

$$(L(\varphi_2(x)),\varphi_1(x)) = \int_0^1 (2-6x+x^2-x^3)(x-x^2)dx = -\frac{3}{20},$$

$$(L(\varphi_3(x)),\varphi_1(x)) = \int_0^1 (6x-12x^2+x^3-x^4)(x-x^2)dx = -\frac{19}{210},$$

$$(L(\varphi_1(x)),\varphi_2(x)) = \int_0^1 (-2+x-x^2)(x^2-x^3)dx = -\frac{3}{20},$$

$$(L(\varphi_2(x)),\varphi_2(x)) = \int_0^1 (2-6x+x^2-x^3)(x^2-x^3)dx = -\frac{13}{105},$$

$$(L(\varphi_3(x)),\varphi_2(x)) = \int_0^1 (6x-12x^2+x^3-x^4)(x^2-x^3)dx = -\frac{79}{840},$$

$$(L(\varphi_1(x)),\varphi_3(x)) = \int_0^1 (-2+x-x^2)(x^3-x^4)dx = -\frac{19}{210},$$

$$(L(\varphi_2(x)),\varphi_3(x)) = \int_0^1 (2-6x+x^2-x^3)(x^3-x^4)dx = -\frac{79}{840},$$

$$(L(\varphi_3(x)),\varphi_3(x)) = \int_0^1 (6x-12x^2+x^3-x^4)(x^3-x^4)dx = -\frac{103}{1260},$$

$$(f(x),\varphi_1(x)) = (-x, x-x^2) = \int_0^1 (x^3-x^2)dx = -\frac{1}{12},$$

$$(f(x),\varphi_2(x)) = \int_0^1 x(x^2-x^3)dx = -\frac{1}{20},$$

$$(f(x),\varphi_3(x)) = \int_0^1 x(x^3-x^4)dx = -\frac{1}{30}.$$

Nomalumlarni aniqlaydigan tenglamalar sistemasini yozamiz:

$$\left.\begin{aligned} \frac{3}{10}c_1+\frac{3}{20}c_2+\frac{19}{210}c_3=\frac{1}{12}, \\ \frac{3}{20}c_1+\frac{13}{105}c_2+\frac{79}{840}c_3=\frac{1}{20}, \\ \frac{19}{210}c_1+\frac{79}{840}c_2+\frac{103}{1260}c_3=\frac{1}{30}, \end{aligned}\right\} \Rightarrow c_1 \cong 0{,}1971,\ \ c_2 \cong 0{,}1473,\ \ c_3 \cong 0{,}0234.$$

Taqribiy analitik yechim

$$y_3(x) = 0{,}1971x(1-x) + 0{,}1473x^2(1-x) + 0{,}0234x^3(x-1) =$$

$$= x(x-1)(0{,}1971 + 0{,}1473x + 0{,}0234x^2)$$

ko'rinishga ega bo'lamiz.

Eslatma. Differensial tenglamadagi $p(x)$, $q(x)$, $f(x)$ funksiyalar murakkab bo'lganda Galyorkin yoki kichik kvadratlar usulidan ko'ra kollokatsiya yoki haydash metodini qo'llab, chegaraviy masalani yechish maqsadga muvofiqroq bo'ladi. Agar $p(x)$, $q(x)$, $f(x)$ lar jadval ko'rinishida berilsa, haydash usulini qo'llagan ma'qul bo'ladi.

Misollar.

Vazifa 1. Chekli ayirmalar usuli bilan chegaraviy masala $h = 0{,}1$ da yechilsin. Hisoblash 10^{-3} aniqlikda olib borilsin.

№ 1. $$y'' + \frac{y'}{x} + 2y = x. \quad \begin{cases} y(0{,}7) = 0{,}5, \\ 2y(1) + 3y'(1) = 1{,}2. \end{cases}$$

№ 2. $$y'' - xy' + 2y = x + 1. \quad \begin{cases} y(0{,}9) - 0{,}5y'(0{,}9) = 2, \\ y(1{,}2) = 1. \end{cases}$$

№ 3. $$y'' + xy' + y = x + 1. \quad \begin{cases} y(0{,}5) + 2y'(0{,}5) = 1, \\ y'(0{,}8) = 1{,}2. \end{cases}$$

№ 4. $$y'' + 2y' - \frac{y}{x} = 3. \quad \begin{cases} y(0{,}2) = 2, \\ 0{,}5y(0{,}5) - y(0{,}5) = 1. \end{cases}$$

№ 5. $$y'' + 2y' - xy = x^2. \quad \begin{cases} y'(0{,}6) = 0{,}7, \\ y(0{,}9) - 0{,}5y'(0{,}9) = 1. \end{cases}$$

№ 6. $$y'' - y' + \frac{2y}{x} = x + 0{,}4. \quad \begin{cases} y(1{,}1) - 0{,}5y'(1{,}1) = 2, \\ y'(1{,}4) = 4. \end{cases}$$

№ 7. $$y'' - 3y' + \frac{y}{x} = 1. \quad \begin{cases} y(0{,}4) = 2, \\ y(0{,}7) + 2y'(0{,}7) = 0{,}7. \end{cases}$$

№ 8. $$y'' + 3y' - \frac{y}{x} = x + 1. \quad \begin{cases} y'(1{,}2) = 1, \\ 2y(1{,}5) - y'(1{,}5) = 0{,}5. \end{cases}$$

№ 9. $y'' - \frac{y'}{2} + 3y = 2x^2.$ $\begin{cases} y(1) + 2y'(1) = 0,6, \\ y(1,3) = 1. \end{cases}$

№ 10. $y'' + 1,5y' - xy = 0,5.$ $\begin{cases} 2y(1,3) - y'(1,3) = 1, \\ y(1,6) = 3. \end{cases}$

№ 11. $y'' + 2xy' - y = 0,4.$ $\begin{cases} 2y(0,3) + y'(0,3) = 1, \\ y'(0,6) = 2. \end{cases}$

№12. $y'' - 0,5xy' + y = 2.$ $\begin{cases} y(0,4) = 1,2, \\ y(0,7) + 2y'(0,7) = 1,4. \end{cases}$

№13. $y'' + \frac{2y'}{x} - 3y = 2.$ $\begin{cases} y'(0,8) = 1,5, \\ 2y(1,1) + y'(1,1) = 3. \end{cases}$

№14. $y'' + 2x^2y' + y = x.$ $\begin{cases} 2y(0,5) - y'(0,5) = 1, \\ y(0,8) = 3. \end{cases}$

№15. $y'' - 3xy' + 2y = 1,5.$ $\begin{cases} y'(0,7) = 1,3, \\ 0,5y(1) + y'(1) = 2. \end{cases}$

№ 16. $y'' + 2xy' - 2y = 0,6.$ $\begin{cases} y'(2) = 1, \\ 0,4y(2,3) - y'(2,3) = 1. \end{cases}$

№17. $y'' + \frac{y'}{x} - 0,4y = 2x.$ $\begin{cases} y(0,6) - 0,3y'(0,6) = 0,6, \\ y'(0,9) = 1,7. \end{cases}$

№18. $y'' - \frac{y'}{2x} + 0,8y = x.$ $\begin{cases} y(1,7) + 1,2y'(1,7) = 2, \\ y'(2) = 1. \end{cases}$

№ 19. $y'' - \frac{y'}{3} + xy = 2.$ $\begin{cases} y(0,8) = 1,6, \\ 3y(1,1) - 0,5y'(1,1) = 1. \end{cases}$

№20. $y'' + 0,8y' - xy = 1,4.$ $\begin{cases} y(1,8) = 0,5, \\ 2y(2,1) + y'(2,1) = 1,7. \end{cases}$

№21. $y'' + 2y' - \frac{y}{x} = \frac{1}{x}.$ $\begin{cases} 0,5y(0,9) + y'(0,9) = 1, \\ y(1,2) = 0,8. \end{cases}$

№22. $y'' - \frac{y'}{4} + \frac{2y}{x} = \frac{x}{2}.$ $\begin{cases} 1,5y(1,3) - y'(1,3) = 0,6, \\ 2y(1,6) = 0,3. \end{cases}$

№23. $y'' - 0,5y' + 0,5xy = 2x.$ $\left\{\begin{array}{l} y'(1) = 0,5, \\ 2y(1,3) - y'(1,3) = 2. \end{array}\right\}$

№24. $y'' + 2y' - 1,5xy = \frac{2}{x}.$ $\left\{\begin{array}{l} y'(0,8) = 1, \\ y(1,1) + 2y'(1,1) = 1. \end{array}\right\}$

№ 25. $y'' + 2xy' - 1,5 = x.$ $\left\{\begin{array}{l} 1,4y(1,1) + 0,5y'(1,1) = 2, \\ y'(1,4) = 2,5. \end{array}\right\}$

№ 26. $y'' - \frac{xy'}{2} + 0,5y = 2x.$ $\left\{\begin{array}{l} 0,4y(0,2) - y'(0,2) = 1,5, \\ y'(0,5) = 0,4. \end{array}\right\}$

№ 27. $y'' + 0,6xy' - 2y = 1.$ $\left\{\begin{array}{l} y(1,5) = 0,6, \\ 2y(1,8) - 0,8y'(1,8) = 3. \end{array}\right\}$

№ 28. $y'' + \frac{y'}{2x} - y = \frac{2}{x}.$ $\left\{\begin{array}{l} y(0,6) = 1,3, \\ 0,5y(0,9) - 1,2y'(0,9) = 1. \end{array}\right\}$

№ 29. $y'' - 0,5x^2y' + 2y = x^2.$ $\left\{\begin{array}{l} y(1,6) + 0,7y'(1,6) = 2, \\ y(1,9) = 0,8. \end{array}\right\}$

№ 30. $y'' - xy' + 2xy = 0,8.$ $\left\{\begin{array}{l} y(1,2) - 0,5y'(1,2) = 1, \\ y'(1,5) = 2. \end{array}\right\}$

Vazifa 2. 1-vazifadagi chegaraviy masalani $h = 0,05$ da haydash usuli bilan yeching.

Vazifa 3. 1-vazifadagi chegaraviy masalani $n = 2$ da kollokatsiya, Galerkin va kichik kvadratlar usullarining birortasi bilan yeching.

IX BOB. XUSUSIY HOSILALI DIFFERENSIAL TENGLAMALARNI TAQRIBIY YECHISH

9.1-§. Umumiy tushunchalar

Xususiy hosilali differensial tenglamalar matematik fizika, gidrodinamika, akustika, texnikaning turli sohalarida keng tatbiqqa ega.

Tabiatda sodir bo'ladigan turli hodisalar, odatda, xususiy hosilali differensial tenglamalar orqali ifodalanadi. Xususiy hosilali differensial tenglamalarni integrallash masalasi oddiy differensial tenglamalarni integrallashga qaraganda ancha murakkabdir. Bunga sabab, birinchidan, xususiy hosilali differensial tenglamalar oddiy differensial tenglamalarga nisbatan har xil va murakkabroq jarayonlarning ifodalanishi bo'lsa, ikkinchidan, boshlang'ich va chegaraviy shartlarning turlicha qo'yilishi hamda masalaning o'lchovidir. Xususiy hosilali differensial tenglamalarning yechimini kamdan-kam hollarda oshkor ko'rinishda chekli formula shaklida topish mumkin bo'ladi. Shuning uchun ham xususiy hosilali differensial tenglamalarni taqribiy yechish metodlari muhim ahamiyatga egadir.

Bayon tushunarli, hisoblash sxemasi va algoritmi soddaroq bo'lishi uchun biz ikki o'zgaruvchili ikkinchi tartibli xususiy hosilali differensial tenglamalarga e'tiborimizni qaratamiz. Tenglamaning tipi va chegaraviy shartlarning xarakteriga qarab har xil masalalar qo'yiladi. Shu munosabat bilan ikkinchi tartibli chiziqli differensial tenglamalar klassifikatsiyasiga to'xtalamiz.

$$
\begin{aligned}
L(u) \equiv a(x,y)\frac{\partial^2 u}{\partial^2 x} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial^2 y} + 2d(x,y)\frac{\partial u}{\partial x} + \\
+ 2e(x,y)\frac{\partial u}{\partial y} + g(x,y)u = f(x,y)
\end{aligned}
\qquad (1)
$$

tenglama D sohada berilgan bo'lsin. Bu yerda $a(x,y)$, $b(x,y)$,..., $g(x,y)$ – koeffitsiyentlar va $f(x,y)$ – ozod had. Bu funksiyalar yopiq $\bar{D}=D\cup\Gamma$ sohada aniqlangan. G – D sohaning chegarasi. Agar D sohada $\delta(x,y)=b^2(x,y)-a(x,y)c(x,y)<0$ bo'lsa, $L(u)=f(x,y)$ tenglama elliptik tipga, $\delta(x,y)>0$ bo'lsa giperbolik tipga, $\delta(x,y)=0$ bo'lsa parabolik tipga ega deyiladi. Tenglamaning tipiga qarab har xil chegaraviy masala qo'yiladi. Masalan, tenglama elliptik tipga oid bo'lsa, chegaraviy shartlar quyidagi ko'rinishda bo'ladi.

Birinchi tur chegaraviy shart:

$$u(x,y)\big|_{\Gamma}=\varphi(M). \tag{2}$$

(1) ning (2) shartni qanoatlantiradigan yechimini topish Dirixle masalasi deyiladi. Bu yerda M – G dagi nuqta.

Ikkinchi tur chegaraviy shart quyidagicha

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma}=\varphi(M) \tag{3}$$

bo'lib, $\frac{\partial u}{\partial n}$ – normal bo'yicha hosila. (1) ning (3) shartni qanoatlantiradigan yechimini topish Neyman masalasi deyiladi.

Uchinchi tur chegaraviy shart esa

$$\left[\alpha(x,y)\frac{\partial u}{\partial n}+\beta(x,y)u\right]_{\Gamma}=\varphi(M) \tag{4}$$

ko'rinishida bo'lib, $\alpha(x,y)$, $\beta(x,y)$ – ma'lum funksiyalar, ular $\left[\alpha^2(x,y)+\beta^2(x,y)\right]_{\Gamma}>0$ shartni bajaradi. $L(u)=f$ tenglamaning (4) ni qanoatlantiradigan yechimini topish masalasi uchinchi chegaraviy masala deyiladi.

Biz quyida xususiy hosilali defferensial tenglamalarni taqribiy yechish usullaridan biri – to'r metodi bayonida asosiy boshlang'ich ma'lumotlarni keltirish bilan chegaralanamiz.

9.2-§. Elliptik tipdagi tenglamalar uchun chegaraviy masalalarni to'r metodi bilan yechish

Bizga (1) ko'rinishidagi tenglama berilgan bo'lsin. Bu tenglamani yuqorida ko'rsatilgan uch turdagi chegaraviy shartlarning birini qo'shib qaraymiz. Chegaraviy masalaning D sohadagi yechimini topish ko'pincha juda murakkab bo'ladi yoki yechimni chekli formula ko'rinishida ifodalab bo'lmaydi. Shuning uchun ham hisoblash amaliyotida yechimni D sohaning barcha nuqtalarida emas, balki D sohaga tegishli ma'lum nuqtalar to'plamida yechimning taqribiy qiymatini topish masalasiga almashtiriladi. Shunday nuqtalar to'plami *to'r* deb nomlanadi. Bunday nuqtalar chekli bo'lib, D sohani taqriban almashtirishi kerak. D_h deb to'r nuqtalarining to'plamini belgilaymiz. (x_i, y_i) D_h ga tegishli nuqta bo'lsin, $u(x_i, y_i)$ esa $u(x, y)$ funksiyaning shu nuqtadagi qiymati. Bunday qiymatlar soni chekli. Ularni topish umuman mumkin va ular differensial tenglama, chegaraviy shart, D soha va uning chegarasi G orqali ifodalanadi. Shu ma'lumotlar asosida differensial tenglamaning chegaraviy shartning xossalarini akslantiradigan va $u(x_i, y_i)$ qiymatni taqriban hisoblash imkonini beradigan munosabatlarini qurish to'r metodining g'oyasidir.

Chegarasi G bo'lgan D soha berilgan bo'lsin. Oxy tekisligida

$$x_i = x_0 + ih, \ \ y_j = y_0 + jh, \ \ (i, j = 0, \pm 1, \pm 2, ...)$$

parallel to'g'ri chiziqlar oilasini o'tkazamiz. Bunda h, l mos ravishda *abssissa va ordinata yo'nalishlaridagi qadamlar* deyiladi. Bu to'g'ri chiziqlarning kesishgan nuqtalari *tugunlar* deyiladi. Tugunlar to'plami esa to'rni tashkil etadi.

Agar ikkita tugun Ox o'qi yoki Oy o'qi bo'ylab, shu yo'nalishda bir-biridan bir qadam uzoqlikda joylashgan bo'lsa, ular *qo'shni tugunlar* deyiladi.

Faqat D sohada yotgan tugunlar to'plamini qaraymiz. Agar biror tugunning to'rttala qo'shni tugunlari to'plamda yotsa, u holda bu tugun *ichki tugun nuqta* deb ataladi. Ichki tugunlar to'plami *to'r soha* deyiladi va D_h orqali belgilanadi. Agar tugunning hech bo'lmaganda bitta qo'shnisi D_h da yotmasa, u holda bu tugun *chegaraviy tugun*, ularning to'plami esa *to'r sohaning chegarasi* deyiladi va Γ_h orqali belgilanadi. Agar D_h va Γ_h to'plamlar birgalikda qaralsa, u holda u *yopiq to'r soha* deyiladi va $\bar{D}_h = D_h \cup \Gamma_h$ orqali belgilanadi.

D_h to'r ustida aniqlangan $u(x,y)$ funksiya uchun $u_{ij} = u(x_i, y_j)$ belgilash kiritamiz va har bir $(i,j) = (x_i, y_j)$ tugun uchun (1) da qatnashadigan hosilalarni bo'lingan ayirmalar orqali ifodalaymiz. Ular quyidagilar:

$$\left(\frac{\partial u}{\partial x}\right)_{(x_i,y_j)} = \frac{u(x_{i+1},y_j)-u(x_i,y_j)}{h} + O(h), \tag{5}$$

$$\left(\frac{\partial u}{\partial x}\right)_{(x_i,y_j)} = \frac{u(x_i,y_j)-u(x_{i-1},y_j)}{h} + O(h), \tag{6}$$

$$\left(\frac{\partial u}{\partial x}\right)_{(x_i,y_j)} = \frac{u(x_{i+1},y_j)-u(x_{i-1},y_j)}{2h} + O(h^2), \tag{7}$$

$$\left(\frac{\partial u}{\partial y}\right)_{(x_i,y_j)} = \frac{u(x_i,y_{j+1})-u(x_i,y_j)}{l} + O(l), \tag{8}$$

$$\left(\frac{\partial u}{\partial x}\right)_{(x_i,y_j)} = \frac{u(x_i,y_j)-u(x_i,y_{j-1})}{l} + O(l), \tag{9}$$

$$\left(\frac{\partial u}{\partial y}\right)_{(x_i,y_j)} = \frac{u(x_i,y_{j+1})-u(x_i,y_{j-1})}{2l} + O(l^2), \tag{10}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_i,y_j)} = \frac{u(x_{i+1},y_j)-2u(x_i,y_j)+u(x_{i-1},y_j)}{h^2} + O(h^2). \tag{11}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{(x_i, y_j)} = \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{l^2} + O(l^2), \qquad (12)$$

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{(x_i, y_j)} = \frac{u(x_{i+1}, y_{j+1}) - u(x_{i-1}, y_{j+1}) - u(x_{i+1}, y_{j-1}) + u(x_{i-1}, y_{j-1})}{4hl} + O(h^2 + l^2), \quad (13)$$

Endi (1) tenglamani (x_i, y_j) nuqtada to'r tenglamaga o'tkazamiz. Quyidagi

$$a(x_i, y_i) = a_{ij}, \quad b(x_i, y_i) = b_{ij}, ..., \; f(x_i, y_i) = f_{ij}$$

belgilashlarni kiritamiz, (7), (10)–(13) formulalarni qo'llab, quyidagiga

$$[L(u)]_{(x_i, y_j)} \equiv \Lambda_h(u(x_i, y_j)) + O(h^2 + l^2) = f_{ij} \qquad (14)$$

ega bo'lamiz, bu yerda

$$\Lambda_h(u(x_i, y_j)) \equiv A_{ij}u(x_i, y_j) + B_{ij}u(x_i, y_j) + D_{ij}u(x_i, y_j) + E_{ij}u(x_i, y_j) +$$
$$+ F_{ij}u(x_i, y_j) + G_{ij}\left(u(x_i, y_j) - u(x_{i-1}, y_{j+1}) - u(x_{i+1}, y_{j-1}) + u(x_{i-1}, y_{j-1})\right)$$

bo'lib,

$$A_{ij} = \frac{a_{ij}}{h^2} + \frac{d_{ij}}{h}; \qquad B_{ij} = \frac{a_{ij}}{h^2} - \frac{d_{ij}}{h}; \qquad D_{ij} = \frac{c_{ij}}{l^2} + \frac{e_{ij}}{l};$$

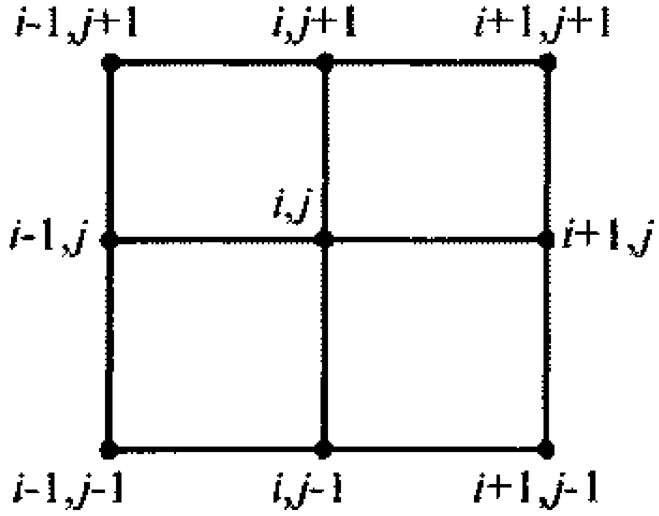
$$\mathrm{E}_{ij} = \frac{c_{ij}}{l^2} - \frac{e_{ij}}{l}; \qquad F_{ij} = -\frac{2a_{ij}}{h^2} - \frac{2c_{ij}}{l^2} + G_{ij}; \qquad G_{ij} = \frac{b_{ij}}{2hl}.$$

$u(x_i, y_i)$ va hokazolar $u(x, y)$ funksiyaning (x_i, y_i) nuqtadagi aniq qiymati. Agar (1) ning yechimi D sohada uzluksiz to'rtinchi tartibgacha hosilaga ega bo'lsa, u holda yetarlicha kichik h va l da (14) dagi $O(h^2 + l^2)$ ni e'tiborga olmasak ham bo'ladi. Unda to'r tenglamani quyidagicha yozish mumkin:

$$\Lambda_h(u_{ij}) = f_{ij} \qquad (15)$$

bu yerda u_{ij} deb, $u(x, y)$ funksiyaning (x_i, y_i) nuqtadagi taqribiy qiymati belgilangan.

Tashlab yuborilgan $O(h^2+l^2)$ had xatolikni bildiradi, ya'ni to'r sohaning ichki nuqtasida $L(u)$ differensial operatorni to'rda aniqlangan $\Lambda(u_{ij})$ operator $O(h^2+l^2)$ aniqlikda approksimatsiya etadi. Approksimatsiya etishga jalb etilgan nuqtalar sxemasi quyidagicha:



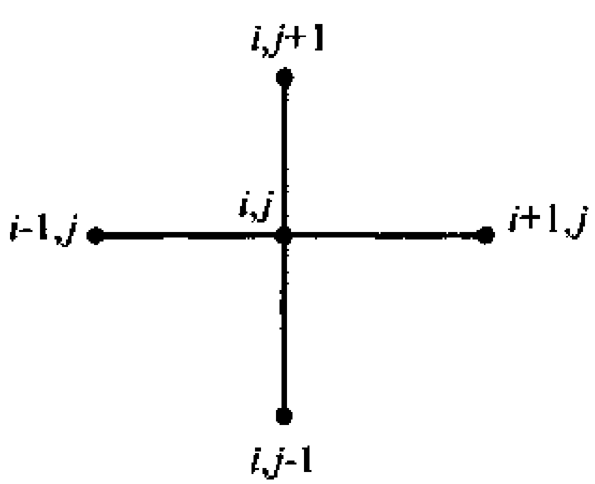
Agar (1) da $b(x,y)\equiv 0$ bo'lsa, unda to'r tenglama

$$L_h(u_{ij})=f_{ij} \qquad (16)$$

ko'rinishida bo'ladi, bu yerda

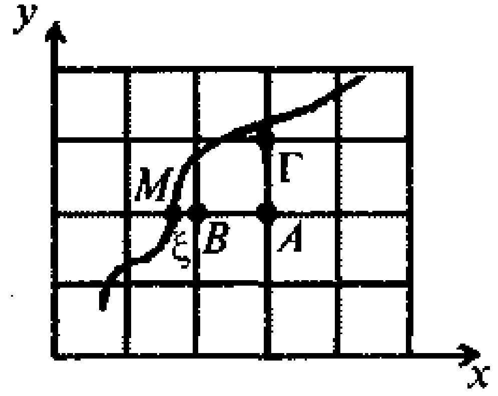
$$L_h(u_{ij})=A_{ij}u_{i+1,j}+B_{ij}u_{i-1,j}+D_{ij}u_{i,j+1}+E_{ij}u_{i,j-1}+F_{ij}u_{ij}$$

va differensial operatorni to'rda aniqlangan operatorga almashtirishda jalb qilinadigan nuqtalar sxemasi esa quyidagi ko'rinishda bo'ladi:



9.3-§. Chegaraviy shartlarni approksimatsiya qilish

Faraz qilaylik, Dirixle masalasini yechayotgan bo'laylik va chegaraviy shart $u|_{\Gamma} = \varphi(M)$ bo'lsin. Qulaylik uchun tomonlari h dan iborat bo'lgan kvadrat to'mi qaraylik:



Bu chizmada $M \in \Gamma$, $B \in \Gamma_h$, $A \in D_h$ bo'lib, $M(x,y)$, $B(x+\delta,y)$, $A(x+\gamma,y)$, $MB = \delta < h$, $\gamma = \delta + h$.

Agar $\delta << h$ bo'lsa, $\varphi(B) = \varphi(M)$ deb olinadi va bu usul chegaraviy shartni to'rning eng yaqin tuguniga ko'chirish deyiladi. Bunda qo'yiladigan xatolikni ko'raylik. Buning uchun $u(B)$ ni Teylor qatoriga yoyamiz:

$$u(B) = u(M) + \frac{\delta}{1!}u'_x(\xi,y) = \varphi(M) + \frac{\delta}{1!}u'_x(\xi,y),$$

bu yerda $x < \xi < x+\delta$. Endi $\delta < h$ ni e'tiborga olsak, oddiy ko'chirishdagi xatolik $o(h)$ bo'ladi, ya'ni $\varphi(B) = \varphi(M)$ deganimizdagi xatolik $o(h)$ bo'lar ekan.

Agar approksimatsiyaga $A \in D_h$ ni ham qatnashtirsak, tabiiyki, $u(B)$ ni hisoblashdagi xatolik kamayadi. Buning uchun quyidagilardan foydalanamiz:

$$u(B) = \varphi(M) + \frac{\delta}{1!}u'_x(M) + \frac{\delta^2}{2!}u''_{xx}(M) + \ldots$$

$$u(A) = \varphi(M) + \frac{\gamma}{1!}u'_x(M) + \frac{\gamma^2}{2!}u''_{xx}(M) + \ldots\surd$$

Bulardan $u'_x(M)$ ni yo‘qotsak, quyidagiga

$$u(B) = \frac{h\varphi(M)+\delta u(A)}{h+\delta} + o(h^2)$$

ega bo‘lamiz. Agar h yetarlicha kichik bo‘lsa, $o(h^2)$ hadni e’tiborga olmasa ham bo‘ladi. Unda

$$u(B) \cong \frac{h\varphi(M)+\delta u(A)}{h+\delta}$$

deyish mumkin. Bu formula ko‘pincha *Kollas formulasi* deyiladi.

9.4-§. To‘r tenglamalar sistemasi yechimining mavjudligi

Faraz qilaylik, bizga chegarasi Γ bo‘lgan Ω sohadagi Dirixle masalasini

$$L(u) = a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial y^2} + c(x,y)\frac{\partial u}{\partial x} + d(x,y)\frac{\partial u}{\partial y} + g(x,y)u = f(x,y) \quad (1)$$

ko‘rinishga ega tenglama va

$$u|_{\Gamma} = \varphi(M) \quad (2)$$

chegaraviy shart uchun yechish kerak bo‘lsin. $\bar{\Omega} = \Omega \cup \Gamma$ da $a(x,y) > 0$, $b(x,y) > 0$ shart bajarilsin, demak, (1) elliptik tipga ega, shuningdek, $\bar{\Omega}$ da $g(x,y) < 0$ bo‘lsin deymiz. To‘g‘ri burchakli to‘rtburchak to‘r olamiz. h va l, mos ravishda, to‘rning qadamlari. Tenglamani to‘r sohaning ichki (x_i, y_i) tugunidagi approksimatsiyasi uchun besh nuqtali sxemani (9.2-§ ga qarang) ishlatamiz. Chegaraviy shartning approksimatsiyasi uchun to‘rning eng yaqin tuguniga ko‘chirish (9.3-§ ga qarang) usulini qo‘llaymiz:

$$L_n(u_{ij}) = A_{ij}u_{i+1j} + B_{ij}u_{i-1j} + C_{ij}u_{ij+1} + D_{ij}u_{ij-1} - E_{ij}u_{ij} = f_{ij} \quad (3)$$

bu yerda $A_{ij}=\frac{a_{ij}}{h^2}+\frac{c_{ij}}{2h}$, $B_{ij}=\frac{a_{ij}}{h^2}-\frac{c_{ij}}{2h}$, $C_{ij}=\frac{b_{ij}}{l^2}+\frac{d_{ij}}{2l}$, $D_{ij}=\frac{b_{ij}}{l^2}-\frac{d_{ij}}{2l}$,

$E_{ij}=-g_{ij}+\frac{2a_{ij}}{h^2}+\frac{2b_{ij}}{l^2}$.

$$u_{ij}|_{\Gamma_h}=\varphi(M^*) \qquad (4)$$

bu yerda $M^*\in\Gamma$ bo'lib, M^* chegaraviy nuqtaga eng yaqin nuqtani bildiradi. (3) tenglama Ω_h to'plamning har bir nuqtasi uchun o'rinli bo'lganligi uchun bunday tenglamalar soni noma'lumlar soniga teng bo'ladi.

(3), (4) formulalar bilan aniqlanadigan chiziqli algebraik tenglamalar sistemasini yechish mumkinligini ko'rsatamiz. Shu bois unga mos quyidagi

$$L_h(v_{ij})=0,\ \ v_{ij}|_{\Gamma}=0 \qquad (5)$$

bir jinsli tenglamalar sistemasi faqat trivial yechimga egaligini ko'rsatamiz. Buning uchun h va l yetarilicha kichik bo'lganda A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij} larni musbat deb hisoblaymiz.

Teorema 1 (maksimum prinsipi). Faraz qilaylik, ω_{ij} lar Ω_h da berilgan qandaydir miqdorlar ($\omega_{ij}\neq const$) va $L_h(\omega_{ij})\geq 0$ shart o'rinli bo'lsin. U holda ω_{ij} lar Ω_h da musbat maksimumga ega bo'la olmaydi. Agar $L_h(\omega_{ij})\leq 0$ shart Ω_h da o'rinli bo'lsa, unda ω_{ij} lar Ω_h da manfiy minimumga ega bo'la olmaydi.

Isbot. Teoremaning birinchi ta'kidining isbotini keltiramiz. $L_h(\omega_{ij})\geq 0$ shart Ω_h da o'rinli bo'lsin, ya'ni ω_{ij} lar Ω_h to'plamda musbat maksimumga erishmasligini ko'rsatish kerak. Teskarisini faraz qilamiz, ya'ni $(x_k,y_k)\in\Omega_h$ uchun $\omega_{ki}=M>0$ bo'lsin. Bu nuqta shundayki, to'rtta qo'shni nuqtalarining kamida birortasida $\omega(x,y)$ funksiyaning qiymati M dan qat'iyan kichik. Unda

$$L_h(\omega_{kl}) = A_{kl}\omega_{k+1,l} + B_{kl}\omega_{k-1l} + C_{kl}\omega_{kl+1} + D_{kl}\omega_{kl-1} - E_{kl}\omega_{kl} <$$
$$< (A_{kl} + B_{kl} + C_{kl} + D_{kl} - E_{kl})M = g_{kl}M \leq 0$$

bo'ladi, chunki A_{kl}, B_{kl}, C_{kl}, D_{kl}, E_{kl} musbat va $g_{kl} < 0$ edi. Bu zidlik teoremani isbotlaydi. Teoremaning ikkinchi ta'kidi shu kabi isbotlanadi.

Isbotlangan teoremaga ko'ra, to'rda aniqlangan ω_{ij} funksiya o'zining musbat maksimumiga va manfiy minimumiga faqatgina Γ_h da erishishi mumkin.

Teorema 2. Agar $g(x,y)$ $\bar{\Omega}$ da musbat bo'lmasa, A_{kl}, B_{kl}, C_{kl}, D_{kl}, E_{kl} lar musbat bo'lsa, u holda (3), (4) tenglamalar sistemasi yechimga ega va yechim yagonadir.

Isbot. (5) tenglamalar sistemasi trivial yechimga egaligini ko'rsatish kifoyadir, ya'ni barcha $v_{ij} \equiv 0$. $L_n(v_{ij}) = 0$ bo'lganligi uchun 1-teoremaning ikkala sharti $L_h(\omega_{ij}) \geq 0$ yoki $L_h(\omega_{ij}) \leq 0$ bajarilgan deyishimiz mumkin. Birinchi ta'kidga ko'ra, v_{ij} lar o'zining musbat maksimumiga Γ_n da erishadi, lekin $v_{ij}|_{\Gamma_n} = 0$ bo'lganligi uchun v_{ij} lar ichida musbati yo'q. Ikkinchi ta'kidga ko'ra, v_{ij} lar ichida manfiy yo'q, degan xulosaga kelamiz. Shuning uchun $v_{ij} \equiv 0$ ayniyatga kelamiz. Demak (5) tenglamalar sistemasi trivial yechimga ega. Bundan (3), (4) tenglamalar sistemasi yechimga ega va uning yagonaligi kelib chiqadi. (3), (4) tenglamalar sistemasini oddiy iteratsiya yoki Zeydel usuli bilan yechish mumkin [7]. Masalan, oddiy iteratsiya usulida (3) ni

$$u_{ij} = \frac{A_{ij}}{E_{ij}}u_{i+1j} + \frac{B_{ij}}{E_{ij}}u_{i-1j} + \frac{C_{ij}}{E_{ij}}u_{ij+1} + \frac{D_{ij}}{E_{ij}}u_{ij} - \frac{f_{ij}}{E_{ij}} \qquad (6)$$

ko'rinishda yozib olish kerak. To'rning qadamlarini shunday olish lozimki, A_{kl}, B_{kl}, C_{kl}, D_{kl}, E_{kl} lar musbat bo'lsin. (6) ning har bir tenglamasining o'ng tomonidagi koeffitsiyentlarning barchasi musbat bo'lib, ularning yig'indisi birdan qat'iy kichik, chunki

$$A_{ij} + B_{ij} + C_{ij} + D_{ij} - g_{ij} = E_{ij},$$

ya'ni

$$A_{ij} + B_{ij} + C_{ij} + D_{ij} < E_{ij}.$$

Bu esa, oddiy iteratsiya usulining yaqinlashuvchi bo'lishligining yetarli shartidir [7].

Shuni ta'kidlash lozimki, murakkabroq to'r tenglamalar sistemasining yechimga egaligini ko'rsatish ancha qiyin va uni hal qilish uchun ancha nozik matematik bilimlarni jalb etish kerak bo'ladi.

9.5-§. Xatolikni baholashda Runge qoidasi

$u(x,y)$ deb 9.4-§dagi (1)–(2) chegaraviy masalaning aniq yechimini, $u_h(x,y)$ deb esa shu masalaning qadamlari h va l bo'lgan to'r metodi bilan taqribiy yechimini belgilaymiz. To'r metodida ko'pincha

$$\varepsilon_h(x,y) = u(x,y) - u_h(x,y)$$

xatolikning h ga nisbatan tartibi ma'lum bo'ladi, ya'ni

$$\varepsilon_h(x,y) = K(x,y)h^p \tag{1}$$

ko'rinishda bo'lib, bu yerda $K(x,y)$ – $\bar{\Omega}$ da musbat chegaralangan qandaydir funksiya, p esa musbat son. Qadam $2h$ bo'lganda

$$\varepsilon_{2h}(x,y) = K(x,y)(2h)^p = 2^p K(x,y)h^p = 2^p \varepsilon_h(x,y) \tag{2}$$

ga ega bo'lamiz. Aniq yechim uchun

$$u(x,y) = u_h(x,y) + \varepsilon_h(x,y),$$

$$u(x,y) = u_{2h}(x,y) + 2^p h(x,y)$$

formulalarga ega bo'lamiz. Bundan

$$\varepsilon_h(x,y) = \frac{u_h(x,y) - u_{2h}(x,y)}{2^p - 1}$$

ko'rinishida bo'lgan xatolikni ifodalaydigan formulaga ega bo'lamiz. (1) va (2) umuman olganda taqribiy xarakterga ega, shuning uchun (3) ham taqribiydir. (3) ning qulayligi shundaki, uni har doim hisoblash mumkin va

$$\tilde{u}_h(x,y) = u_h(x,y) + \frac{u_h(x,y) - u_{2h}(x,y)}{2^p - 1}$$

qiymat $u_h(x,y)$ dan aniqroq bo'lishini kutish mumkin. Amaliyotda quyidagicha ish tutiladi: taqribiy yechimni berilgan tugunlarda h va $2h$ qadamlar bilan hisoblab, $u_h(x,y)$ va $u_{2h}(x,y)$ qiymatlar taqqoslanadi. Agar ular berilgan aniqlikda ustma-ust tushsa, u holda taqribiy yechim $u_h(x,y)$ deb olinadi. Aks holda h qadamni ikkiga bo'lib, $u_{\frac{h}{2}}(x,y)$ qiymat hisoblanadi, so'ngra aniqlikning yetarliligini bilish uchun yuqoridagidek ish tutiladi.

Agar chegaraviy shart Kollas formulasi bo'yicha approksimatsiya qilingan bo'lsa, 9.4-§dagi (1)–(2) chegaraviy masalani yechishdagi xatolikning tartibi h ga nisbatan $p = 2$ bo'ladi. Bu holda

$$\varepsilon_h(x,y) = \frac{u_n(x,y) - u_{2h}(x,y)}{3}$$

bo'ladi.

9.6-§. Parabolik tipdagi differensial tenglamalarni to'r usuli bilan yechish

Faraz qilaylik, $\Gamma = \{0 < x < 1, 0 < t < T\}$ sohada ushbu

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t) \qquad (1)$$

parabolik tenglamaning (issiqlik o'tkazuvchanlik tenglamasining)

$$u(x,0) = \varphi(x) \qquad (2)$$

boshlang'ich shart va

$$u(0,t)=\psi_0(t) \quad u(1,t)=\psi_1(t) \tag{3}$$

chegaraviy shartlarni qanoatlantiradigan yechimini topish talab qilinsin. Bu yerda $\varphi(x)$, $\psi_0(t)$, $\psi_1(t)$ – berilgan funksiyalar. (1)–(3) masalaning yechimi mavjud, yagona yechim $u(x,t)$ esa kerakli tartibgacha hosilalarga ega deb hisoblaymiz.

Ayirmali sxema qurish uchun Γ sohani x va t koordinatalar bo'yicha mos ravishda $h=\frac{1}{N}$, $\tau=\frac{T}{K}$ qadamli to'g'ri to'rtburchak to'r bilan almashtiramiz. (x_i,t_j), $i=0,1,...,N$, $j=0,1,...,K$ nuqtalar to'plamini *to'r sohaning tugunlari* deymiz.

Quyidagi

$$(x_i,t_0),\ i=0,1,...,N,\ (x_0,t_j) \text{ va } (x_N,t_j),\ j=0,1,...,K$$

tugunlari to'r sohaning *chegaraviy tugunlari*, qolganlari esa to'r sohaning *ichki tugunlari* deyiladi.

Endi (1) ni (x_i,t_j) nuqtada approksimatsiya qilish uchun $\frac{\partial u}{\partial t}$ va $\frac{\partial^2 u}{\partial x^2}$ hosilalarni

$$\left(\frac{\partial u}{\partial t}\right)_{(x_i,t_j)} \cong \frac{u_{ij+1}-u_{ij}}{\tau}, \tag{4}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_i,t_j)} \cong \frac{u_{i+1j}-2u_{ij}+u_{i-1j}}{h^2} \tag{5}$$

taqribiy formulalar bilan almashtiramiz. (4) va (5) ni (1) ga qo'yamiz, hamda (2) va (3) ning approksimatsiyasini yozamiz, natijada quyidagi ayirmali masalani hosil qilamiz:

$$\frac{u_{ij+1}-u_{ij}}{\tau}=\frac{u_{i+1j}-2u_{ij}+u_{i-1j}}{h^2}+f_{ij},\quad i=1,2,...,N-1,\ j=0,1,...,K-1 \tag{6}$$

$$u_{i0}=\varphi_i,\quad i=0,1,...,N \tag{7}$$

$$u_{0j}=\psi_{0j},\ u_{Nj}=\psi_{1j},\quad j=0,1,...,K-1\ . \tag{8}$$

(6), (7), (8) chiziqli algebraik tenglamalar sistemasi bo'lib, tenglamalar soni noma'lumlar soniga tengdir. Agar j-qatlamdagi yechimning qiymatlari ma'lum bo'lsa, $(j+1)$-qatlamdagi yechimning qiymati

$$u_{ij+1} = u_{ij} + \frac{\tau}{h^2}(u_{i+1j} - 2u_{ij} + u_{i-1j}) + \tau f_{ij}, \quad i = 1,2,\ldots,N-1 \qquad (9)$$

formula bilan aniqlanadi. u_{0j+1} va u_{Nj+1} lar mos ravishda ψ_{0j+1} va ψ_{1j+1} ga teng. Shuning uchun (6), (7), (8) sxema oshkor deyiladi. Quyida ayirmali sxemaning turg'unligi va yaqinlashishi uchun zaruruiy shartni chiqaramiz. Xususan, (6), (7), (8) sxemani $\tau \le 0,5h^2$ shart o'rinli bo'lsagina qo'llash mumkinligini ko'rsatamiz. Buning uchun (6) ga mos

$$\frac{u_{kj+1} - u_{kj}}{\tau} = \frac{u_{k+1j} - 2u_{kj} + u_{k-1j}}{h^2} \qquad (10)$$

birjinsli tenglamani ko'ramiz. (10) ning xususiy yechimini

$$u_{kj}^{(\varphi)} = q^j e^{ikh\varphi} \qquad (11)$$

ko'rinishida izlaymiz, bu yerda i – mavhum bir, φ – ixtiyoriy haqiqiy son va q – aniqlanishi lozim bo'lgan noma'lum son. (11) ni (10) ga qo'yib va $e^{ikh\varphi}$ ga qisqartirib

$$\frac{q-1}{\tau} = \frac{e^{ih\varphi} - 2 + e^{-ih\varphi}}{h^2}$$

ni hosil qilamiz, bundan

$$q = 1 - 4\alpha \sin^2 \frac{h\varphi}{2}, \quad \alpha = \frac{\tau}{h^2} \qquad (12)$$

ekanligini topamiz. (11) ko'rinishdagi yechim uchun mos $u_{k0}(\varphi) = e^{ikh\varphi}$ boshlang'ich shartlar chegaralangan. Agar φ ning qandaydir qiymatida (11) dagi q moduli bo'yicha birdan katta bo'lsa, u holda $j \to \infty$ da (11) cheksiz o'suvchi bo'ladi. Bunday holda (10) ayirmali tenglama noturg'un deyiladi, chunki yechimning boshlang'ich shartlarga uzluksiz bog'liqligi buziladi. Agar $|q| \le 1$

bo'lsa, (11) ko'rinishdagi yechimlar j ning ixtiyoriy qiymatida chegaralangan bo'lganligi uchun (10) ayirmali tenglama turg'un deyiladi. (10) tenglama uchun $|q| \leq 1$ shart ixtiyoriy φ uchun faqat $\alpha \leq 0,5$ bo'lgandagina bajariladi. Demak, (6), (7), (8) ayirmali sxemani $\tau \leq 0,5h^2$ shart bajarilgandagina qo'llash mumkin. Bunday ayirmali sxema shartli turg'un deyiladi.

Endi oshkormas sxemani ko'ramiz. Buning uchun (x_i, t_j), (x_{i+1}, t_{j+1}), (x_i, t_{j+1}), (x_{i-1}, t_{j+1}) nuqtlarni (1) ni approksimatsiya qilish uchun jalb etamiz va natijada

$$\frac{u_{ij+1}-u_{ij}}{\tau}=\frac{u_{i+1\,j+1}-2u_{ij+1}+u_{i-1\,j+1}}{h^2}+f_{ij+1},\ \ i=1,2,...,N-1,\ j=0,1,...,K-1$$
$$u_{i0}=\varphi_i,\quad i=0,1,...,N, \tag{13}$$
$$u_{0j+1}=\psi_{0j+1},\ u_{Nj+1}=\psi_{1j+1},\quad j=0,1,...,K-1.$$

ko'rinishidagi oshkormas sxemaga ega bo'lamiz. Bu sxema τ bo'yicha birinchi, h bo'yicha ikkinchi tartibli approksimatsiyaga ega. (13) ni quyidagicha yozamiz:

$$\alpha u_{i+1\,j+1}-(1+2\alpha)u_{ij+1}+\alpha u_{i+1\,j-1}+u_{ij}=\tau f_{ij+1},$$
$$i=1,2,...,N-1,\ j=0,1,...,K-1 \tag{14}$$
$$u_{0j+1}=\psi_{0j+1},\ u_{Nj+1}=\psi_{1j+1},\ \ j=0,1,...,K-1.$$

(14) chiziqli algebraik tenglamalar sistemasi, uning matritsasi uch diagonalli, uni haydash usuli bilan yechish mumkin, chunki diagonal elementlari salmoqli.

(14) ayirmali sxema turg'unligining zaruriy shartini chiqaramiz. Buning uchun

$$\frac{u_{ij+1}-u_{ij}}{\tau}=\frac{u_{i+1\,j+1}-2u_{ij+1}+u_{i-1\,j+1}}{h^2}$$

bir jinsli tenglamaning xususiy yechimini (11) ko'rinishda izlaymiz va natijada

$$q=\left(1+4\alpha\cdot\sin^2\frac{h\varphi}{2}\right)^{-1}, \quad \alpha=\frac{\tau}{h^2}$$

ga ega bo'lamiz. Demak, ixtiyoriy φ, τ, h larda $|q|\leq 1$, ya'ni (13) ayirmali sxema absolut turg'un. Absolut turg'un sxemaning afzalligi shundaki, to'r qadamlariga hech qanaqa shartning yo'qligidir. Bu, o'z navbatida, hisoblashdagi talab qilingan aniqlikni ta'min qilish uchun h va τ larni tanlash imkonini beradi.

9.7-§. Giperbolik tipdagi tenglamalarni to'r metodi bilan yechish

Bizga

$$L(u)\equiv a(x,y)\frac{\partial^2 u}{\partial x^2}-b(x,y)\frac{\partial^2 u}{\partial y^2}+c(x,y)\frac{\partial u}{\partial x}+d(x,y)\frac{\partial u}{\partial y}+g(x,y)u=f(x,y) \quad (1)$$

ko'rinishidagi tenglama berilgan bo'lsin, bu yerda $a,\ b,\ c,\ d,\ g,\ f$ – ma'lum biror G sohada ikki marta uzluksiz differensiallanuvchi funksiyalar, $u(x,y)$ esa topilishi lozim funksiya. G sohada $a(x,y)\ b(x,y)>0$ shart o'rinli deymiz, ya'ni (1) giperbolik tipga ega. Bundan tashqari, aniqlik uchun $a(x,y)$, $b(x,y)$ G da musbat bo'lsin deb hisoblaymiz.

Quyidagi masalalarni ko'ramiz.

Koshi masalasi: $G=\{y>0,-\infty<x<\infty\}$ sohada ikki marta uzluksiz differensiallanuvchi shunday $u(x,y)$ funksiya topilsinki, G sohada (1) tenglamani qanoatlantirib, $y=0$ to'g'ri chiziqda

$$u\big|_{y=0}=\varphi(x), \quad \left.\frac{\partial u}{\partial y}\right|_{y=0}=\psi(x) \qquad (2)$$

boshlang'ich shartlarni qanoatlantirsin, bu yerda $\varphi(x)$, $\psi(x)$ – berilgan ma'lum funksiyalar.

Aralash chegaraviy masala: $G=\{0<y<Y,\alpha<x<\beta\}$ sohada ikki marta uzluksiz differensiallanuvchi shunday $u(x,y)$ funksiyani topilsinki, u G da (1) tenglamani qanoatlantirib, $y=0$ to'g'ri chiziqda (2) boshlang'ich shartni va $x=\alpha$, $x=\beta$ to'g'ri chiziqda quyidagi uch turdagi chegaraviy shartlarni birortasini qanoatlantirsin:

A) birinchi tur chegaraviy shartlar:

$$u\big|_{x=\alpha}=\Phi_\alpha(y),\quad u\big|_{x=\beta}=\Phi_\beta(y); \tag{3}$$

B) ikkinchi tur chegaraviy shartlar:

$$\frac{\partial u}{\partial x}\bigg|_{x=\alpha}=\Psi_\alpha(y),\quad \frac{\partial u}{\partial x}\bigg|_{x=\beta}=\Psi_\beta(y); \tag{4}$$

D) uchinchi tur chegaraviy shartlar:

$$\left.\begin{aligned}\left[\sigma_0(y)\frac{\partial u}{\partial x}+\sigma_1(y)u\right]_{x=\alpha}&=\Gamma_\alpha(y),\\ \left[\tau_0(y)\frac{\partial u}{\partial x}+\tau_1(y)\,u\right]_{x=\beta}&=\Gamma_\beta(y),\end{aligned}\right\} \tag{5}$$

bu yerda Φ,Ψ,Γ berilgan funksiyalar va σ_0, σ_1, τ_0, τ_1 lar

$$|\delta_0(y)|+|\delta_1(y)|>0 \quad \text{va} \quad |\tau_0(y)|+|\tau_1(y)|>0$$

shartlarni qanoatlantiradi.

1. Koshi masalasini yechish.

(1), (2) Koshi masalasini to'r metodi bilan yechish masalasini ko'ramiz. Qadamlari h va l bo'lgan to'g'ri to'rtburchak to'r olamiz:

$$G_{hl}=\{x_i=ih,\, i=0,\pm1,\pm2,\dots,\, h>0,\ y_j=jl,\ j=0,1,2,\dots,\, l>0\}$$

va (1) tenglamani to'r sohaning ichki (x_i,y_i) tugunida approksimatsiya etish uchun (x_i,y_j), $(x_{i\pm1},y_j)$, $(x_i,y_{j\pm1})$ nuqtalarni jalb qilamiz.

Natijada quyidagi

$$\begin{gathered}L_n(u_{ij})\equiv A_{ij}u_{ij+1}+B_{ij}u_{ij-1}+C_{ij}u_{i+1j}+D_{ij}u_{i-1j}+E_{ij}u_{ij}=f_{ij},\\ i=0,\pm1,\pm2,\dots;\quad j=1,2,\dots,\end{gathered} \tag{6}$$

ko'rinishidagi to'r tenglamalar sistemasiga ega bo'lamiz, bu yerda

$$A_{ij}=-\frac{b_{ij}}{l^2}+\frac{d_{ij}}{2l},\ B_{ij}=-\frac{b_{ij}}{l^2}-\frac{d_{ij}}{2l},\ C_{ij}=\frac{a_{ij}}{h^2}+\frac{c_{ij}}{2h},$$

$$D_{ij}=\frac{a_{ij}}{h^2}-\frac{c_{ij}}{2h},\quad E_{ij}=-\frac{2a_{ij}}{h^2}+\frac{2b_{ij}}{l^2}+g_{ij}.$$

Agar (1)ning yechimi qaralayotgan sohada x va y o'zgaruvchilar bo'yicha to'rtinchi tartibgacha hosilalari uzluksiz va chegaralangan bo'lsa, u holda (1)ni (6)ga o'tkazishdagi xatolik $R_{ij}(u)=o\left(h^2+l^2\right)$ ko'rinishida bo'ladi.

(1) boshlang'ich shartlarni

$$u_{io}=\varphi_i,\ \ u_{i1}=l\psi_1+u_{i0},\ i=0,\pm 1,\pm 2,\ldots \tag{7}$$

ko'rinishidagi to'r funksiyalar bilan approksimatsiya qilamiz. Ikkinchi boshlang'ich shart approksimatsiyasining xatoligi $r_i(u)=o(l)$ bo'lishligi ayondir. Shunday qilib, (1), (2) differensial masala (6), (7) to'r masalaga o'tkazildi.

(7) formula u_{ij} to'r funksiyaning $j=0$ (nolinchi qatlam)da va $j=1$ (birinchi qatlam)da qiymatlarini topish imkonini beradi. $j>1$ bo'lgandagi u_{ij} ning qiymatlarini esa (6) formula bilan aniqlanadi. Bunda l shunday bo'lishi kerakki $A_{ij}<0$ ligiga erishish zarur.

Endi $\frac{l}{h}=\alpha$ qanday bo'lishligini aniqlaymiz. Bu savolga to'liq javob olish maqsadida quyidagi Koshi masalasini ko'ramiz:

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0, \tag{8}$$

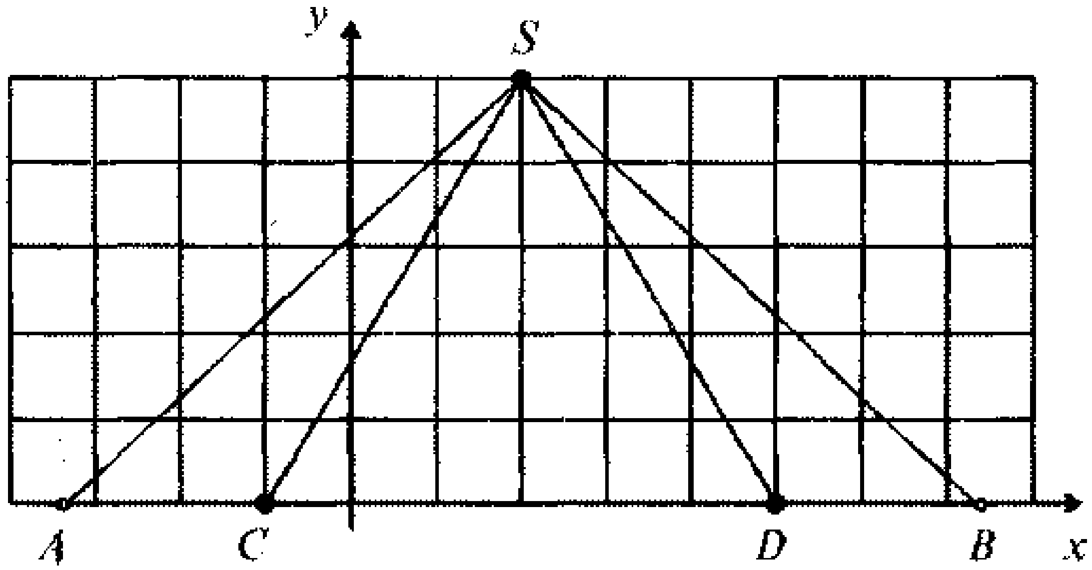
$$u\big|_{y=0}=\varphi(x),\ \ \frac{\partial u}{\partial y}\bigg|_{y=0}=\Psi(x). \tag{9}$$

Bu yerda $G=\{\,y>0,\ -\infty<x<\infty\,\}$.

Bu holda (6) to'r tenglama

$$L_n(u_{ij}) = -\frac{1}{l^2}u_{i\,j+1} - \frac{1}{l^2}u_{i\,j-1} + \frac{1}{h^2}u_{i+1\,j} + \frac{1}{h^2}u_{i-1\,j} + \left(\frac{2}{l^2} - \frac{2}{h^2}\right)u_{ij} = 0 \qquad (10)$$

ko'rinishida bo'ladi, (7) esa o'zgarishsiz qoladi. Koordinatalari (x_i, y_i) bo'lgan S nuqtada (8), (9) masala yechimining qiymatini hisoblash talab qilingan bo'lsin.



Ma'lumki, (8) tenglamaning S nuqtadagi yechimining qiymati (x_i, y_j) nuqtadan o'tuvchi

$$y - y_j = x - x_i, \quad y - y_j = -x + x_i$$

xarakteristikalar $y = 0$ to'g'ri chiziqda ajratadigan kesmadagi shartlar bilan, ya'ni AB kesmadagi boshlang'ich shartlar bilan bir qiymatli aniqlanadi. (8) tenglamaning xarakteristikalari o'zaro perpendikular bo'lib, Ox o'qi bilan 45° va 135° burchaklarni tashkil etadi. ASB uchburchak (8) differensial tenglamaning aniqlanganlik uchburchagi deyiladi. Agar to'r funksiyaning S nuqtadagi yechimi u_{ij} ni (10) formula yordamida hisoblasak, u boshlang'ich shartni CD kesmadagi qiymatlari orqali ifodalanadi. Bu kesma S nuqtadan o'tuvchi va Ox o'qi bilan $\angle SCD = \text{arctg}\alpha$ va $\angle SDB = \text{arctg}(-\alpha)$ tashkil etuvchi to'g'ri chiziqlar hosil qilgan uchburchak CSD ning asosidir. Bu uchburchak (10) ayirmali tenglamaning aniqlanganlik uchburchagi deyiladi. Yuqoridagi chizmada $\angle SAD < \angle SCD$, $\text{tg}\angle SCD = \alpha = \frac{l}{h} > 1$ bo'lgan hol keltirilgan. Bunday hol, ya'ni $\alpha = \frac{l}{h} > 1$ quyidagi sababga ko'ra yaroqsizdir. Agar boshlang'ich shartlarni AC va DB kesmalarda o'zgartirsak, (8), (9) differensial masalaning yechimi G sohada, jumladan, S nuqtada o'zgarishi kerak. Ammo (10), (7) ayirmali masalaning yechimi esa o'zgarmay qoladi. Demak, $\alpha > 1$ bo'lganida (10), (7) ayirmali masalaning yechimi $h \to 0, \tau \to 0$da (8), (9) Koshi masalasi yechimiga yaqinlashmaydi. Shuning uchun to'r sohaning qadamlari nisbati shunday bo'lishi kerakki, $\alpha \leq 1$ bo'lsin, ya'ni ΔASB ΔCSD ning ichida bo'lishi kerak. Shuni eslatamizki, umumiy holda differensial tenglamaning aniqlangan uchburchagi egri chiziqli uchburchakdan iborat bo'ladi, ammo bu uchburchak ayirmali tenglamaning aniqlanganlik uchburchagi ichida yotishi lozim.

Endi chegaraviy masalani to'r usuli bilan yechish masalasini qaraymiz. Faraz qilaylik, (1) tenglamaning (2) boshlang'ich shartlarni va (5) chegaraviy shartlarni qanoatlantiruvchi yechimini topish masalasi berilgan bo'lsin. Berilgan $G = \{0 < y < Y, \alpha < x < \beta\}$ sohani qadamlari h va l bo'lgan to'r bilan qoplaymiz. To'rning ichki tugunlarida tenglamaning approksimatsiyasini, chegaraviy tugunlarida esa (2) va (5) shartlarning approksimatsiyasini qilamiz. Tenglama $o(h^2 + l^2)$ xatolik bilan (2) boshlang'ich shart $o(h)$ chegaraviy shartlar $o(l)$ xatolik bilan approksimatsiya qilingan bo'ladi. (1) ning approksimatsiyasi uchun $(x_i, y_i), (x_{i\pm 1}, y_i), (x_i, y_{i\pm 1})$ tugunlar jalb etilgan. Natijada quyidagi ayirmali tenglamalar sistemasi hosil bo'ladi:

$$L_n(u_{ij}) \equiv A_{ij}u_{i\,j+1} + B_{ij}u_{i\,j-1} + C_{ij}u_{i+1\,j} + D_{ij}u_{i-1\,j} + E_{ij}u_{ij} = f_{ij}, \quad i = 0,1,2,\ldots,N, \quad j = 0,1,\ldots,M \ (Ml = Y) \tag{11}$$

$$u_{i0} = \varphi_i, \; u_{i1} = l\psi_i + u_{i0}, \; i = 0,1,\ldots,N \qquad (12)$$

$$\left.\begin{aligned} &\delta_{0j}\frac{u_{ij}-u_{0j}}{h} + \delta_{1j}u_{0j} = \Gamma_{\alpha j}, \\ &\tau_{0j}\frac{u_{Nj}-u_{N-1j}}{h} + \tau_{1j}u_{Nj} = \Gamma_{\beta j}, \quad j = 0,1,\ldots,M \end{aligned}\right\} \qquad (13)$$

Bu (11), (12), (13) to'r masala (1), (2), (5) chegaraviy masalani tenglama bo'yicha $o(l^2 + h^2)$ boshlang'ich shartlarni $o(h)$, chegaraviy shartlarni esa $o(l)$ xatolik bilan approksimatsiya qiladi. To'r masalani qatlamlar bo'yicha yechish mumkin. Haqiqatan ham, $j = 0$ va $j = 1$ bo'lganda (12) bilan u_{i0}, u_{i1} $i = 0,1,\ldots,N$ lar topiladi. So'ng, l ni tanlash hisobiga $A_{ij} = -\frac{b_{ij}}{l^2} + \frac{d_{ij}}{2l} < 0$ bo'lishligini ta'minlab, (11) formuladan foydalanib $u_{12}, u_{22}, \ldots, u_{N-12}$ larning qiymatini aniqlaymiz, (13) dan esa u_{02}, u_{N2} aniqlanadi. Shunday qilib, $j = 2$ da, ya'ni ikkinchi qatlamda u_{ij} larning qiymatlari barcha tugunlarda aniqlanadi. Keyingi qatlamlardagi tugunlarda ham u_{ij} ning qiymatlari shu kabi aniqlanadi.

Bobga tegishli tayanch so'zlar: to'r soha, ichki nuqta, chegaraviy nuqta, to'r funksiya, approksimatsiya xatoligi, oshkor sxema, oshkormas sxema, turg'unlik, ayirmali tenglamaning aniqlanganlik uchburchagi.

Savollar va topshiriqlar

1. Ikki o'zgaruvchili ikkinchi tartibli xususiy hosilali differensial tenglamalar klassifikatsiyasi.
2. Elliptik tipdagi tenglamaga qo'yiladigan chegaraviy shartlar turlari.
3. To'r sohani qurish.

4. Qo‘shni tugunlar tushunchasi.

5. To‘r sohaning ichki tugunlari.

6. To‘r sohaning chegaraviy tugunlari.

7. Yopiq to‘r soha.

8. Birinchi tartibli hosilalarni bo‘lingan ayirmalar orqali ifodalaydigan formulalarni chiqaring va qadamga nisbatan xatolik tartibini ko‘rsating.

9. Ikkinchi tartibli hosilalarni bo‘lingan ayirmalar orqali ifodalaydigan formulalarni chiqaring va qadamga nisbatan xatolik tartiblarini ko‘rsating.

10. $\Lambda_h(u_{ij}) = f_{ij}$ to‘r tenglamani hosil qilishda qatnashadigan tugunlar sxemasini ko‘rsating.

11. $L_h(u_{ij}) = f_{ij}$ to‘r tenglamani hosil qilishda ishtirok etuvchi tugunlar sxemasini ko‘rsating.

12. Chegaraviy shartlarni approksimatsiya etish usullarini chiqaring, qadamga nisbatan xatolik tartibini ko‘rsating.

13. Maksimum prinsipi.

14. To‘r tenglamalar sistemasining yechimi mavjudligi.

15. Xatolikni baholashda Runge qoidasi.

16. Parabolik tipdagi tenglama uchun qo‘yilgan chegaraviy masalani yozing.

17. Ayirmali masalani chiqaring.

18. Qanday sxema oshkor deyiladi?

19. Oshkormas sxemani chiqaring.

20. Oshkor sxemaning turg‘unligining zaruriy shartini chiqaring.

21. Oshkormas sxemada hosil bo‘lgan tenglamalar sistemasini yechish qanday bajariladi?

22. Qanday sxemalar shartli turg‘un, absolut turg‘un deb nomlanadi?

23. Oshkormas sxema turg'unligining zaruriy shartini chiqaring.

24. Giperbolik tipdagi tenglamalar uchun Koshi masalasi.

25. Chegaraviy masalalar turlari.

26. Koshi masalasini to'r usuli bilan yechish.

27. Qadamlar nisbati $\frac{l}{h} = \alpha$ qanday bo'lishining tahlili.

28. Chegaraviy masalani to'r usuli bilan yechish.

Misol 1. $G = \{0 < x < 1,\ 0 < t < 0{,}025\}$ sohada

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

$$u(x,0) = \sin \pi x, \tag{2}$$

$$u(0,t) = u(1,t) = 0 \tag{3}$$

differensial masalani to'r metodi bilan $h = 0{,}1$, $\alpha = 0{,}5$ bo'lganda oshkor sxemadan foydalanib yeching.

Yechish. $\alpha = 0{,}5$ bo'lganligi uchun $\tau = 0{,}005$ bo'ladi. Berilgan masalani ayirmali masalaga o'tkazamiz, u quyidagicha:

$$u_{i,j+1} = \frac{u_{i-1,j} + u_{i+1,j}}{2}, \quad i = 1,2,\ldots,9, \quad j = 0,1,2,3,4, \tag{1'}$$

$$u_{i0} = \sin \pi x_i = \sin \pi \cdot 0{,}1i, \quad i = 1,2,\ldots,10, \tag{2'}$$

$$u_{0j} = u_{10,j} = 0, \quad j = 0,1,\ldots,5. \tag{3'}$$

Jadvalga boshlang'ich va chegaraviy qiymatlarni yozamiz. Ularning simmetriyasidan foydalanib jadvalni faqat $x = 0;\ 0{,}1;\ 0{,}2;\ 0{,}3;\ 0{,}4;\ 0{,}5$ lar uchun to'ldiramiz. Yechimni birinchi qatlamdagi qiymatlarini (1') da $j = 0$ desak,

$$u_{i,1} = \frac{u_{i-1,0} + u_{i+1,0}}{2}$$

formula bilan hisoblaymiz. Bu qiymatlarni jadvalning ikkinchi yo'liga joylashtiramiz.

Endi (1') da $j = 2$ deb,

$$u_{i,2} = \frac{u_{i-1,1} + u_{i+1,1}}{2}$$

formula bilan yechimning ikkinchi qatlamdagi qiymatlarini topamiz va jadvalning uchinchi yo‘liga yozib qo‘yamiz. Shu tariqa t ning $0,005; 0,010; 0,015; 0,020; 0,025$ qiymatlari uchun yechimning qiymatlarini aniqlaymiz.

j	t \ x	0	0,1	0,2	0,3	0,4	0,5
0	0	0	0,3090	0,5878	0,8090	0,9511	1,0000
1	0,005	0	0,2939	0,5590	0,7699	0,9045	0,9511
2	0,010	0	0,2795	0,5316	0,7318	0,8602	0,9045
3	0,015	0	0,2658	0,5056	0,6959	0,8182	0,8602
4	0,020	0	0,2528	0,4808	0,6619	0,7780	0,8182
5	0,025	0	0,2404	0,4574	0,6294	0,7400	0,7780

Misol 2. Quyidagi

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad (1)$$

$$u(x,0) = 4x(1-x), \qquad (2)$$

$$u(0,t) = u(1,t) = 0, \qquad (3)$$

chegaraviy masalani haydash usuli bilan $h = 0,1$, $l = 0,01$ deb yeching.

Yechish. $h = 0,1$, $l = 0,01$ bo‘lganligi uchun $\alpha = \frac{l}{h^2} = 1$ bo‘ladi.

Berilgan chegaraviy masalani ayirmali masalaga o‘tkazamiz:

$$u_{i+1,j+1} - 3u_{ij+1} + u_{i-1,j+1} + u_{ij} = 0, \; i = 1, 2, \ldots, 10; \; j = 0, 1, 2, \ldots, \qquad (1')$$

$$u_{i0} = 4x_i(1 - x_i), \qquad (2')$$

$$u_{0j} = u_{10j} = 0. \qquad (3')$$

Haydash usuliga ko‘ra (1') ni

$$u_{ij+1} = a_{ij+1}(b_{ij+1} + u_{i+1,j+1}) \qquad (4)$$

ko‘rinishiga keltiriladi, bu yerda a_{ij+1}, b_{ij+1} miqdorlar quyidagicha ketma-ket aniqlanadi:

$$a_{1,j+1} = \frac{1}{2+\alpha} = \frac{1}{3}, \; b_{1,j+1} = u_{1,j}, \qquad (5)$$

$$\left.\begin{aligned} a_{i,j+1} &= \frac{1}{3-a_{i-1,j+1}} \\ b_{ij+1} &= a_{i-1,j+1}b_{i-1,j+1} + u_{ij} \end{aligned}\right\} \qquad (6)$$

Keyin (3') dan $u_{10,j+1} = 0$ ligi ma’lum bo‘lganligi uchun (4) dan u_{ij+1} larni $i = 9, 8, \ldots, 1$ deb topiladi.

Shunday qilib, $u(x,t)$ ni t_j dagi qiymatlari ma’lum bo‘lsa, $t = t_{j+1}$ dagi qiymatlarini haydash usuli bilan topish mumkin. Hisoblash natijalarini jadvalda akslantiramiz. Jadvalning birinchi yo‘lida $u_{i0}(i = 0, 1, \ldots, 10)$ larni yozamiz, (5) formula bo‘yicha $a_{11} = 0{,}3333$, $b_{11} = u_{10} = 4x_1(1 - x_1) = 0{,}36$.

Endi (6) da $j = 0$ deb

$$a_{21} = \frac{1}{3-a_{11}} = \frac{3}{8} = 0{,}3750\;;\; b_{21} = a_{11}b_{11} + u_{20} = 0{,}12 + 0{,}64 = 0{,}760$$

$$a_{31} = \frac{1}{3-a_{21}} = 0{,}3810;\; b_{31} = a_{21}b_{21} + u_{30} = 0{,}3810 \cdot 0{,}7600 + 0{,}84 = 1{,}1250$$

va hokazo.

a_{i1} va b_{i1} larni jadvalga yozamiz. Bu hisoblashlar haydash usulining to‘g‘ri yo‘li deyiladi. Haydash usulining teskari usuli quyidagicha:

chegaraviy shartdan

$$u_{10,1} = 0.$$

$u_{i1}\left(i = 9,8,\ldots,1\right)$ larni (4) formula yordamida hisoblaymiz:

$u_{91} = a_{91}\left(b_{91} + u_{10,1}\right) = 0,3818 \cdot 0,8120 = 0,3100,$

$u_{81} = a_{81}\left(b_{81} + u_{9,1}\right) = 0,3818 \cdot \left(1,1860 + 0,3100\right) = 0,5712,$

... ...

$u_{11} = a_{11}\left(b_{11} + u_{21}\right) = 0,3333 \cdot \left(0,36 + 0,5712\right) = 0,3100.$

i	0	1	2	3	4	5	6	7	8	9	10
u_{i0}	0	0,3600	0,6400	0,8400	0,9600	1,0000	0,9600	0,8400	0,6400	0,3600	0
a_{i1}	0	0,3333	0,3750	0,3810	0,3818	0,3818	0,3818	0,3818	0,3818	0,3812	0
b_{i1}	0	0,3600	0,7600	1,1250	1,3890	1,5300	1,5440	1,5440	1,1860	0,8120	0
u_{i1}	0	0,3100	0,5715	0,7640	0,8820	0,9210	0,8820	0,8820	0,5715	0,3100	0

Shunday qilib, birinchi qatlamda joylashgan tugunlarda yechimning taqribiy qiymatlari topildi. Keyingi qatlamlardagi tugunlarda yechimni aynan shu amallarni bajarib topiladi.

Misol 3. Quyidagi

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

$$u(x,0) = 0,2 \cdot x \cdot (1-x) \cdot \sin(\pi x), \quad \frac{\partial u}{\partial y}\Big|_{y=0} = 0, \qquad (2)$$

$$u(0,y) = 0, \quad u(1,y) = 0. \qquad (3)$$

Chegaraviy masalani to'r metodi bilan $h = l = 0,1$ deb yeching.

Yechish. Differensial masalani ayirmali masalaga o'tkazamiz.

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}, \qquad (1')$$

$$u_{i0}=0,2\cdot x_i\cdot(1-x_i)\cdot\sin(\pi x_i),\quad \frac{u_{i1}-u_{i,-1}}{2l}=0, \tag{2'}$$

$$u_{oj}=0,\quad u_{10j}=0. \tag{3'}$$

Ayirmali tenglama (1') ni $j=0$ da yozamiz:

$$u_{i1}=u_{i+1,0}+u_{i-1,0}-u_{i,-1},$$

Bundan va (2') ning ikkinchisidan $u_{i,-1}$ ni yo'qotsak,

$$u_{i1}=\frac{u_{i+1,0}-u_{i-1,0}}{2}$$

hosil bo'ladi. Natijada quyidagi

$$\left.\begin{array}{l} u_{i,j+1}=u_{i+1,j}+u_{i-1,j}-u_{i,j-1},\quad j=1,2,...,4 \\ u_{i0}=0,2\cdot x_i\cdot(1-x_i)\cdot\sin(\pi x_i),\quad u_{i1}=\frac{u_{i+1,0}-u_{i-1,0}}{2},\quad i=1,2,...,9 \\ u_{oj}=0,\quad u_{10j}=0,\quad j=0,1,...5 \end{array}\right\} \tag{4}$$

ayirmali tenglamalar sistemasiga ega bo'lamiz. Hisoblash natijalarini jadvalga joylashtiramiz. Jadvalning birinchi yo'liga u_{i0} larning qiymatlarini yozamiz, ikkinchi yo'liga esa birinchi yo'ldagi qiymatlardan foydalanib, u_{i1} larning qiymatlarini joylashtiramiz. Keyingi qatlamdagilarni (1') dan foydalanib aniqlaymiz. Masalaning simmetriyasini e'tiborga olib, natijani faqat $0\le x\le 0,5$ oraliq uchun keltiramiz.

$y_j \backslash x_i$	0	0,1	0,2	0,3	0,4	0,5
0	0	0,0056	0,0188	0,0340	0,0457	0,0500
l	0	0,0094	0,0198	0,0323	0,0420	0,0457
$2l$	0	0,0142	0,0229	0,0278	0,0323	0,0340
$3l$	0	0,0135	0,0222	0,0229	0,0198	0,0189
$4l$	0	0,0080	0,0135	0,0142	0,0095	0,0056
$5l$	0	0	0	0,0001	0	0,0001

Misol 4. Quyidagi

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + x + y, \quad (1)$$

$$u(x,0) = (1{,}5 \cdot x^2 + 1{,}2)\sin(\pi x), \quad \frac{\partial u}{\partial y}\Big|_{y=0} = 0{,}1x, \quad (2)$$

$$u(0,y) = u(1,y) = 0 \quad (3)$$

differensial masalani to‘r usuli bilan $h = l = 0{,}1$ bo‘lganda yeching.

Yechish. Bu masalani ayirmali masalaga o‘tkazamiz:

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}, +0{,}01(x_i + y_j), \quad (1')$$

$$I = 1, 2, \ldots, 9, \ j = 1, 2, 3, 4;$$

$$u_{i0} = (1{,}5 \cdot x_i^2 + 1{,}2)\sin(\pi \cdot x_i), \quad u_{i1} = u_{i0} + 0{,}1 \cdot 0{,}1 \cdot x_i, \quad (2')$$

$$I = 1, 2, \ldots, 10;$$

$$u_{0j} = u_{10j} = 0, \ j = 0,1,2,3,4,5. \quad (3')$$

Natijalarni jadval ko‘rinishida keltiramiz. Jadvalning 1-ustunida chegaraviy shartlarni yozamiz, birinchi yo‘lda (2') ning birinchisidan u_{i0} larni aniqlab yozamiz, ikkinchi yo‘lga esa (2') ning ikkinchisidan topilgan u_{i1} ning qiymatlarini yozamiz. So‘ng (1') da $j = 1$ deb uchinchi yo‘lni to‘ldiramiz. $j = 2, 3, 4$ dagi qiymatlar (1') dan shunday aniqlanadi.

y_j / x_i	0	0,1	0,2	0,3	0,4	0,5
0	0	0,3754	0,7406	1,0800	1,3696	1,5750
l	0	0,3764	0,7426	1,0830	1,3736	1,5800
$2l$	0	0,3702	0,7228	1,0412	1,2994	1,1792
$3l$	0	0,3514	0,6738	0,9452	0,8538	1,0268
$4l$	0	0,3086	0,5798	0,4934	0,6816	0,5374
$5l$	0	0,2344	0,1352	0,3242	0,1860	0,3464

Misollar.

Vazifa 1. $G = \{0 < x < 0,6,\ 0 < t < 0,01\}$ sohada

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

$$u(x,0) = f(x),\ u(0,t) = \varphi(t),\ u(0,6,t) = \psi(t).$$

differensial masalani to'r metodi bilan $h = 0,1$, $\alpha = 1/6$ bo'lganda oshkor sxemadan foydalanib yeching.

№ 1. $u(x,0) = \cos 2x$, $u(0,t) = 1 - 6t$, $u(0,6,t) = 0,3624$.

№ 2. $u(x,0) = x(x+1)$, $u(0,t) = 0$, $u(0,6,t) = 2t + 0,96$.

№ 3. $u(x,0) = 1,2 + \lg(x + 0,4)$, $u(0,t) = 0,8 + t$, $u(0,6,t) = 1,2$.

№ 4. $u(x,0) = \sin 2x$, $u(0,t) = 2t$, $u(0,6,t) = 0,932$.

№ 5. $u(x,0) = 3x(2 - x)$, $u(0,t) = 0$, $u(0,6,t) = t + 2,52$.

№ 6. $u(x,0) = 1 - \lg(x + 0,4)$, $u(0,t) = 1,4$, $u(0,6,t) = t + 1$.

№ 7. $u(x,0) = \sin(0,55x + 0,03)$, $u(0,t) = t + 0,03$, $u(0,6,t) = 0,354$.

№ 8. $u(x,0) = 2x(1 - x) + 0,2$, $u(0,t) = 0,2$, $u(0,6,t) = t + 0,68$.

№ 9. $u(x,0) = \sin x + 0,08$, $u(0,t) = 0,08 + 2t$, $u(0,6,t) = 0,6446$.

№ 10. $u(x,0) = \cos(2x + 0,19)$, $u(0,t) = 0,932$, $u(0,6,t) = 0,1798$.

№ 11. $u(x,0) = 2x(x + 0,2) + 0,4$, $u(0,t) = 2t + 0,4$, $u(0,6,t) = 1,36$.

№ 12. $u(x,0) = \lg(x + 0,26) + 1$, $u(0,t) = 0,415 + t$, $u(0,6,t) = 0,9345$.

№ 13. $u(x,0) = \sin(x + 0,45)$, $u(0,t) = 0,435 - 2t$, $u(0,6,t) = 0,8674$.

№ 14. $u(x,0) = 0,3 + x(x + 0,4)$, $u(0,t) = 0,3$, $u(0,6,t) = 6t + 0,9$.

№ 15. $u(x,0) = (x - 0,2)(x + 1) + 0,2$, $u(0,t) = 6t$, $u(0,6,t) = 0,84$.

№ 16. $u(x,0) = x(0,3 + 2x)$, $u(0,t) = 0$, $u(0,6,t) = 6t + 0,9$.

№ 17. $u(x,0)=\sin(x+0{,}48)$, $u(0,t)=0{,}4618$, $u(0{,}6,t)=3t+0{,}882$.

№ 18. $u(x,0)=\sin(x+0{,}02)$, $u(0,t)=3t+0{,}02$, $u(0{,}6,t)=0{,}581$.

№ 19. $u(x,0)=\cos(x+0{,}48)$, $u(0,t)=6t+0{,}887$, $u(0{,}6,t)=0{,}4713$.

№ 20. $u(x,0)=\lg(2{,}63-x)$, $u(0,t)=3(0{,}14-t)$, $u(0{,}6,t)=0{,}3075$.

№ 21. $u(x,0)=1{,}5-x(1-x)$, $u(0,t)=3(0{,}5-t)$, $u(0{,}6,t)=1{,}26$.

№ 22. $u(x,0)=\cos(x+0{,}845)$, $u(0,t)=6(t+0{,}11)$, $u(0{,}6,t)=0{,}1205$.

№ 23. $u(x,0)=\lg(2{,}42+x)$, $u(0,t)=0{,}3838$, $u(0{,}6,t)=6(0{,}08-t)$.

№ 24. $u(x,0)=0{,}6+x(0{,}8-x)$, $u(0,t)=0{,}6$, $u(0{,}6,t)=3(0{,}24+t)$.

№ 25. $u(x,0)=\cos(x+0{,}66)$, $u(0,t)=3t+0{,}79$, $u(0{,}6,t)=0{,}3058$.

№ 26. $u(x,0)=\lg(1{,}43+2x)$, $u(0,t)=0{,}1553$, $u(0{,}6,t)=3(t+0{,}14)$.

№ 27. $u(x,0)=0{,}9+2x(1-x)$, $u(0,t)=3(0{,}3-2t)$, $u(0{,}6,t)=1{,}38$.

№ 28. $u(x,0)=\lg(1{,}95+x)$, $u(0,t)=0{,}29-6t$, $u(0{,}6,t)=0{,}4065$.

№ 29. $u(x,0)=2\cos(x+0{,}55)$, $u(0,t)=1{,}705$, $u(0{,}6,t)=0{,}817+3t$.

№ 30. $u(x,0)=x(1-x)+0{,}2$, $u(0,t)=0{,}2$, $u(0{,}6,t)=2(t+0{,}22)$.

Vazifa 2. $G=\{0\le x\le 1,\ 0\le t\le 0{,}5\}$ sohada

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2},$$

$$u(x,0)=f(x),\quad \frac{\partial u}{\partial t}\Big|_{t=0}=\Phi(x),$$

$$u(0,t)=\varphi(t),\quad u(1,t)=\psi(t).$$

differensial masalani to‘r usuli bilan $h=l=0{,}1$ bo‘lganda yeching.

№ 1. $f(x)=x(x+1)$, $\Phi(x)=\cos x$, $\varphi(t)=0$, $\psi(t)=2(t+1)$.

№ 2. $f(x)=x\cos\pi x$, $\Phi(x)=x(2-x)$, $\varphi(t)=2t$, $\psi(t)=-1$.

№ 3. $f(x)=\cos\frac{\pi x}{2}$, $\Phi(x)=x^2$, $\varphi(t)=1+2t$, $\psi(t)=0$.

№ 4. $f(x)=(x+0{,}5)(x-1),\ \Phi(x)=\sin(x+0{,}2),\ \varphi(t)=t-0{,}5,\ \psi(t)=3t$.

№ 5. $f(x)=2x(x+1)+0{,}3,\ \Phi(x)=2\sin x,\ \varphi(t)=0{,}3,\ \psi(t)=4{,}3+t$.

№ 6. $f(x)=(x+0{,}2)\sin\frac{\pi x}{2},\ \Phi(x)=1+x^2,\ \varphi(t)=0,\ \psi(t)=1{,}2(t+1)$.

№ 7. $f(x)=x\sin\pi x,\ \Phi(x)=(x+1)^2,\ \varphi(t)=2t,\ \psi(t)=0$.

№ 8. $f(x)=3x(1-x),\ \Phi(x)=\cos(x+0{,}5),\ \varphi(t)=2t,\ \psi(t)=0$.

№ 9. $f(x)=x(2x-0{,}5),\ \Phi(x)=\cos 2x,\ \varphi(t)=t^2,\ \psi(t)=1{,}5$.

№ 10. $f(x)=(x+1)\sin\pi x,\ \Phi(x)=x^2+x,\ \varphi(t)=0,\ \psi(t)=0{,}5t$.

№ 11. $f(x)=(1-x)\cos\frac{\pi x}{2},\ \Phi(x)=2x+1,\ \varphi(t)=2t+1,\ \psi(t)=0$.

№ 12. $f(x)=0{,}5x(x+1),\ \Phi(x)=x\cos x,\ \varphi(t)=2t^2,\ \psi(t)=1$.

№ 13. $f(x)=0{,}5(x^2+1),\ \Phi(x)=x\sin 2x,\ \varphi(t)=0{,}5+3t,\ \psi(t)=1$.

№ 14. $f(x)=(x+1)\sin\frac{\pi x}{2},\ \Phi(x)=1-x^2,\ \varphi(t)=0{,}5t,\ \psi(t)=2$.

№ 15. $f(x)=x^2\cos\pi x,\ \Phi(x)=x^2(x+1),\ \varphi(t)=0{,}5t,\ \psi(t)=t-1$.

№ 16. $f(x)=(1-x^2)\cos\pi x,\ \Phi(x)=2x+0{,}6,\ \varphi(t)=1+0{,}4t,\ \psi(t)=0$.

№ 17. $f(x)=(x+0{,}5)^2,\ \Phi(x)=(x+1)\sin x,\ \varphi(t)=0{,}5(0{,}5+t),\ \psi(t)=2{,}25$.

№ 18. $f(x)=1{,}2x-x^2,\ \Phi(x)=(x+0{,}6)\sin x,\ \varphi(t)=0,\ \psi(t)=0{,}2+0{,}5t$.

№ 19. $f(x)=(x+0{,}5)(x+1),\ \Phi(x)=\cos(x+0{,}3),\ \varphi(t)=0{,}5,\ \psi(t)=3-2t$.

№ 20. $f(x)=0{,}5(x+1)^2,\ \Phi(x)=(x+0{,}5)\cos\pi x,\ \varphi(t)=0{,}5,\ \psi(t)=2-3t$.

№ 21. $f(x)=(x+0{,}4)\sin\pi x,\ \Phi(x)=(x+1)^2,\ \varphi(t)=0{,}5t,\ \psi(t)=0$.

№ 22. $f(x)=(2-x)\sin\pi x,\ \Phi(x)=(x+0{,}6)^2,\ \varphi(t)=0{,}5t,\ \psi(t)=0$.

№ 23. $f(x)=x\cos\frac{\pi x}{2},\ \Phi(x)=2x^2,\ \varphi(t)=0,\ \psi(t)=t^2$.

№ 24. $f(x)=(x+0,4)\cos\frac{\pi x}{2}$, $\Phi(x)=0,3(x^2+1)$, $\varphi(t)=0,4$, $\psi(t)=1,2t$.

№ 25. $f(x)=(1-x^2)+x$, $\Phi(x)=2\sin(x+0,4)$, $\varphi(t)=1$, $\psi(t)=(t+1)^2$.

№ 26. $f(x)=0,4(x+0,5)^2$, $\Phi(x)=x\sin(x+0,6)$, $\varphi(t)=0,1+0,5t$, $\psi(t)=0,9$.

№ 27. $f(x)=(x+0,5)^2\cos\pi x$, $\Phi(x)=(x+0,7)^2$, $\varphi(t)=0,5$, $\psi(t)=2t-1,5$.

№ 28. $f(x)=(x+2)(0,5x+1)$, $\Phi(x)=2\cos\left(x+\frac{\pi}{6}\right)$,

$\varphi(t)=2$, $\psi(t)=4,5-3t$.

№ 29. $f(x)=(x^2+1)(1-x)$, $\Phi(x)=1-\sin x$, $\varphi(t)=1$, $\psi(t)=0,5t$.

№ 30. $f(x)=(x+0,2)\sin\frac{\pi x}{2}$, $\Phi(x)=1+x^2$, $\varphi(t)=0,6t$, $\psi(t)=1,2$.

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